NOMINAL STRESS EFFECTS ON THE SIZE AND SHAPE OF PLASTIC ZONES

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ABSTRACT

Proper calculation of the linear stress field in an Inglis plate with a very sharp notch, or in a cracked infinite plate using the Westergaard function, shows that the traditional estimates for size and shape of the plastic zone **pz** ahead of a crack tip, obtained after *assuming* that they are controlled only by the crack stress intensity factor, significantly *underestimate* **pz**.

KEYWORDS

Plastic zone estimates, nominal stress effects, Inglis and Westergaard stress functions.

INTRODUCTION

The plastic zone $\mathbf{pz(\theta)}$ size and shape ahead of a crack tip are traditionally estimated using simplified linear elastic (LE) stress fields, which depend only on the stress intensity factor (SIF) $\mathbf{K_I}$ and on the distance from the tip. E.g., the Mises elastic-plastic frontier is given by [1]:

$$\begin{aligned} pz(\theta)_{pl-\sigma} &= (K_I^2/2\pi S_Y^2)\cdot\cos^2(\theta/2)\cdot[1+3\sin^2(\theta/2)] \\ pz(\theta)_{pl-\varepsilon} &= (K_I^2/2\pi S_Y^2)\cdot\cos^2(\theta/2)\cdot[(1-2\nu)^2+3\sin^2(\theta/2)] \end{aligned} \tag{1}$$

where $\mathbf{S_Y}$ is the yielding strength, θ is the angle measured from the crack plane, ν is Poisson's coefficient, and **x** and **y** are Cartesian coordinates whose origin is the crack tip. But the Irwin solution obtained from the Westergaard stress function is only valid very near to the tip:

$$\sigma_y = \sigma\cdot(x+a)/\sqrt{[(x+a)^2 - a^2]} \cong \sigma a/\sqrt{(2ax)} = \sigma\sqrt{(\pi a)}/\sqrt{(2\pi x)} = K_I/\sqrt{(2\pi x)} \text{ if } x \to 0 \tag{2}$$

Thus, the complete stress field in a cracked piece *cannot* be generated form $\mathbf{K_I}$ only. E.g., a cracked infinite plate tensioned by a nominal stress σ_n obviously has $\sigma_y(x \to \infty, y = 0) = \sigma_n$, not $\sigma_y(x \to \infty, 0) = 0$, as it would be wrongly predicted assuming that σ_y depends only on $\mathbf{K_I}$. If $\kappa = K_I/\sqrt{(2\pi r)}$, a first estimate for the σ_n effect on the **pz** shape and size in the cracked plate can be obtained forcing the Williams stress to obey $\sigma_y(x \to \infty, 0) = \sigma_n$, to obtain e.g.:

$$\sigma(\theta)_{Mises,\sigma-pl} = [(\kappa f_x)^2 + (\kappa f_y + \sigma_n)^2 - (\kappa f_x)(\kappa f_y + \sigma_n) + 3(\kappa f_{xy})^2]^{1/2} \tag{3}$$

where $\mathbf{f_x}$, $\mathbf{f_y}$ and $\mathbf{f_{xy}}$ describe respectively the σ_x, σ_y and τ_{xy} angular θ-dependence in mode I. Making $\sigma_{Mises,\sigma-pl} = \mathbf{S_Y}$, and repeating this process for plane strain, the plastic zones $\mathbf{pz(\theta)_{\sigma-pl}}$ and $\mathbf{pz(\theta)_{\varepsilon-pl}}$ can be finally estimated. The results plotted in Fig.1 show that the σ_n influence on **pz** can be indeed quite large, since it is not uncommon to find yielding safety factors as low as $\phi_Y = S_Y/\sigma_n = 1.25$ (or nominal stresses as high as $\sigma_n = 0.8S_Y$) in practical structures.

PLASTIC ZONE ESTIMATES BY INGLIS

Equation (3) gives an idea of the error associated with the classical $\mathbf{pz(K_I)}$ estimates, but it is not a mathematically sound solution for this problem. A better estimate can be generated by the Inglis stress field in an infinite plate with a sharp elliptical notch whose semi-axis **a** >> **b** is perpendicular to σ_n, where **b** is the smaller ellipsis semi-axis. In elliptical coordinates (α, β), the notch is simply described by $\alpha = \alpha_0$, where:

$$\mathbf{x = c\cdot\cosh\alpha\cdot\cos\beta,\ y = c\cdot\sinh\alpha\cdot\sin\beta,\ a = c\cdot\cosh\alpha_0,\ b = c\cdot\sinh\alpha_0,}\ \text{and}\ \mathbf{c = a/\cos\alpha_0} \tag{4}$$

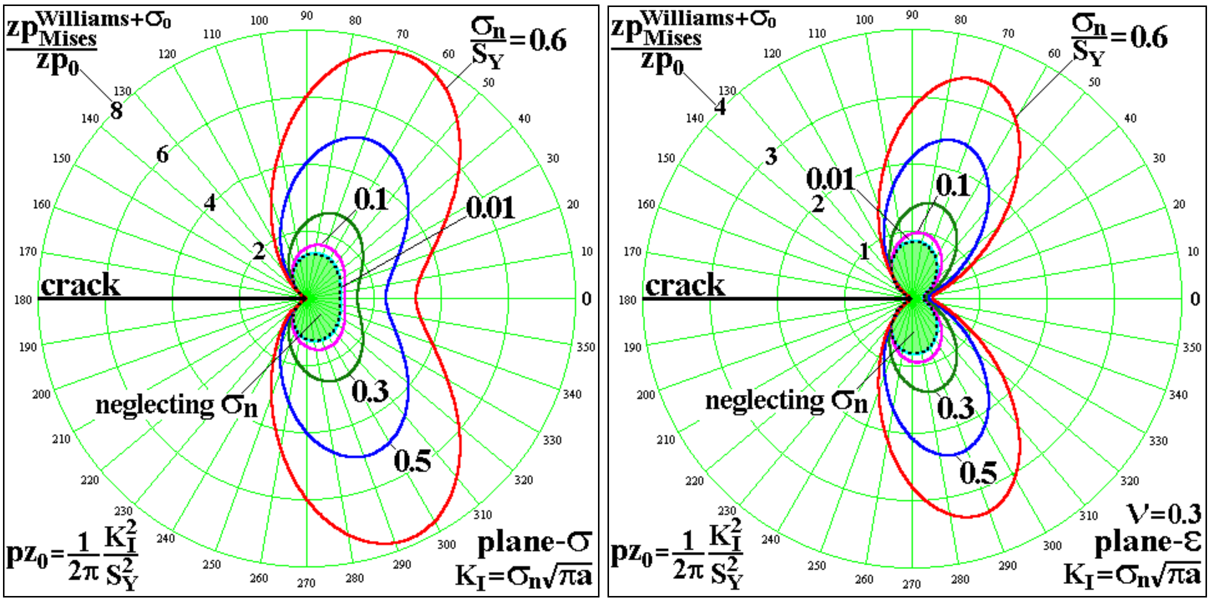

Fig.1: Estimates for $\mathbf{pz(\theta)_{\sigma\text{-}pl}}$ and $\mathbf{pz(\theta)_{\varepsilon\text{-}pl}}$ obtained by summing σ_n to the Williams $\mathbf{\sigma_y(K_I)}$ stress, to force $\sigma_y \to \sigma_n$ far from the crack tips in an infinite plate, where $\mathbf{K_I = \sigma_n\sqrt{(\pi a)}}$.

The linear elastic stresses in the Inglis plate loaded by a general bi-axial nominal stress field are given by [2]:

$$\begin{cases}
\sigma_\alpha = \dfrac{1}{(\cosh 2\alpha - \cos 2\beta)^2}\displaystyle\sum_n A_n\{(n+1)e^{(1-n)\alpha}\cos(n+3)\beta + (n-1)e^{-(n+1)\alpha}\cos(n-3)\beta - \\
\quad -[4e^{-(n+1)\alpha} + (n+3)e^{(3-n)\alpha}]\cos(n+1)\beta + [4e^{(1-n)\alpha} + (3-n)e^{-(n+3)\alpha}]\cos(n-1)\beta\} + \\
\quad +B_n\{e^{-(n+1)\alpha}[n\cos(n+3)\beta + (n+2)\cos(n-1)\beta] - [(n+2)e^{(1-n)\alpha} + ne^{-(n+3)\alpha}]\cos(n+1)\beta\} \\
\sigma_\beta = \dfrac{1}{(\cosh 2\alpha - \cos 2\beta)^2}\displaystyle\sum_n A_n\{(3-n)e^{(1-n)\alpha}\cos(n+3)\beta - (n+3)e^{-(n+1)\alpha}\cos(n-3)\beta - \\
\quad -[4e^{-(n+1)\alpha} - (n-1)e^{(3-n)\alpha}]\cos(n+1)\beta + [4e^{(1-n)\alpha} + (n+1)e^{-(n+3)\alpha}]\cos(n-1)\beta\} - \\
\quad -B_n\{e^{-(n+1)\alpha}[n\cos(n+3)\beta + (n+2)\cos(n-1)\beta] - [(n+2)e^{(1-n)\alpha} + ne^{-(n+3)\alpha}]\cos(n+1)\beta\} \\
\tau_{\alpha\beta} = \dfrac{1}{(\cosh 2\alpha - \cos 2\beta)^2}\displaystyle\sum_n A_n\{(n-1)e^{(1-n)\alpha}\sin(n+3)\beta + (n+1)e^{-(n+1)\alpha}\sin(n-3)\beta - \\
\quad - (n+1)e^{(3-n)\alpha}]\sin(n+1)\beta - (n-1)e^{-(n+3)\alpha}]\sin(n-1)\beta\} - \\
\quad -B_n\{e^{-(n+1)\alpha}[n\sin(n+3)\beta + (n+2)\sin(n-1)\beta] - [(n+2)e^{(1-n)\alpha} + ne^{-(n+3)\alpha}]\sin(n+1)\beta\}
\end{cases} \tag{5}$$

Fortunately, only 5 constants of these series are non-zero if the plate is loaded by a uniaxial tensile nominal stress σ_n perpendicular to the semi-axis **a** of its elliptical hole:

$$\mathbf{A_1 = -\sigma_n(1 + 2e^{2\alpha 0})/16,\ A_{-1} = \sigma_n/16,\ B_1 = \sigma_n e^{4\alpha 0}/8,\ B_{-1} = \sigma_n(1 + \cosh 2\alpha_0)/4}\ \text{and}\ \mathbf{B_{-3} = \sigma_n/8} \tag{6}$$

Assuming that in this case the plate has a very sharp elliptical notch that is similar to a crack whose tip radius is given by ρ = $\mathbf{b^2/a}$ = **CTOD/2** = $\mathbf{2K_I^2/\pi E'S_Y}$, where **CTOD** is the crack tip opening displacement, $\mathbf{K_I = \sigma_n\sqrt{(\pi a)}}$, and **E' = E** in plane stress or $\mathbf{E' = E/(1 - \nu^2)}$ in plane strain, then:

$$\mathbf{K_t = 1 + 2\cdot\frac{a}{b} = 1 + 2\sqrt{\frac{a}{\rho}} = 1 + 2\cdot\sqrt{\frac{a\pi E'S_Y}{2\cdot\sigma_n^2\pi a}} \Rightarrow \frac{a}{b} = \sqrt{\frac{E'}{2\cdot\sigma_n}\cdot\frac{S_Y}{\sigma_n}} = \sqrt{\frac{E'\phi_Y}{2\cdot\sigma_n}}} \tag{7}$$

Using the semi-axis ratio given by (7) in (4) to obtain $\alpha_0 = \mathbf{tanh^{-1}(b/a)}$ and substituting this value in (6), the stress in the cracked plate can be calculated by (5). This task may be tedious, but it is not particularly difficult. Using the resulting stresses σ_α, σ_β and $\tau_{\alpha\beta}$, the elastic-plastic frontier of the plane stress plastic zone can be estimated by:

$$\sigma_{\mathbf{Mises},\sigma-\mathbf{pl}} = \sqrt{\sigma_\alpha^2 + \sigma_\beta^2 - \sigma_\alpha\sigma_\beta + 3\tau_{\alpha\beta}^2} = \mathbf{S_Y} \tag{8}$$

This equation can be numerically solved for α and β by first fixing one of the variables and then finding the other that makes $\sigma_{\mathbf{Mises}} = \mathbf{S_Y}$. To obtain these points in polar coordinates, as shown in Fig.2, they can be first transformed into Cartesian coordinates **(x, y)** using (4) and then to polar coordinates $(\mathbf{r}, \theta)$. In this way, the effect of $\sigma_\mathbf{n}$ on the size and shape of the calculated plastic zone can be visually evaluated by plotting $\mathbf{pz(\theta)/pz_0}$, where $\mathbf{pz_0} = \mathbf{K_I^2}/(2\pi\mathbf{S_Y^2})$ is the traditional LEFM estimate obtained by equating the yield strength to the stress normal to the crack in its plane, $\sigma_\mathbf{y}(\mathbf{r}, \theta = \mathbf{0}) = \mathbf{K_I}/\sqrt{(2\pi\mathbf{r})} = \mathbf{S_Y}$.
This exercise can be repeated for the plane strain case by calculating:

$$\sigma_{\mathbf{Mises},\varepsilon-\mathbf{pl}} = \sqrt{\mathbf{0.5}[(\sigma_\alpha - \sigma_\beta)^2 + (\sigma_\alpha - \sigma_z)^2 + (\sigma_z - \sigma_\beta)^2] + 3\tau_{\alpha\beta}^2} = \mathbf{S_Y} \tag{9}$$

where $\sigma_\mathbf{z} = \nu(\sigma_\alpha + \sigma_\beta)$. As shown in Fig. 2, the effect of the nominal stress $\sigma_\mathbf{n}$ on the Inglis plastic zone, although a little less than that estimated by the approximation shown in Fig.1, is indeed significant. It is worth to emphasize that this so-called Inglis **pz** has been rigorously calculated using the Mises yield criterion and the *exact* Inglis solution for the linear elastic stress in a cracked plate, when the crack is modeled by an elliptical sharp notch of tip radius $\rho = \mathbf{CTOD/2}$, a quite reasonable hypothesis.

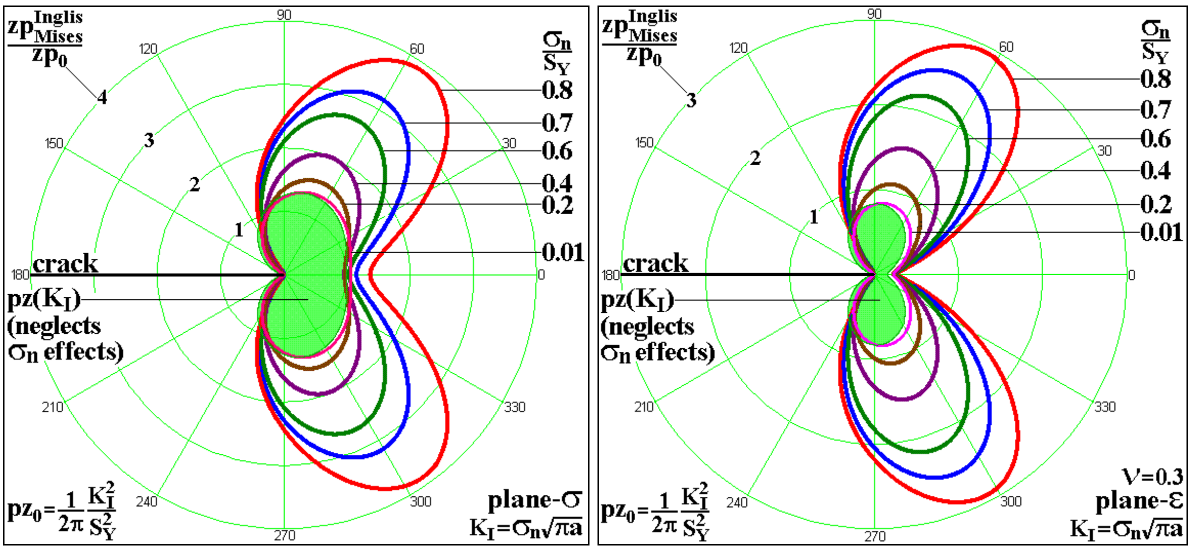

Fig.2: Plastic zones calculated using Mises and the Inglis stress field in an infinite cracked plate tensioned by a nominal stress σ_n perpendicular to the crack, modeling the crack as a very sharp elliptical notch of tip radius ρ = **CTOD/2**, in plane stress and in plane strain.

PLASTIC ZONE ESTIMATES BY WESTERGAARD

The Westergaard stress function provides an alternate way to rigorously estimate the size and the shape of the plastic zones ahead of the crack tips in an infinite plate loaded by a tensile stress $\sigma_\mathbf{n}$ perpendicular to the crack whose length is **2a**. However, since the elastic-plastic **pz** frontier is *not* adjacent to the crack tip, its size and shape should not be calculated supposing, as usual, that the Irwin simplification used to obtain $\mathbf{K_I} = \sigma_n\sqrt{(\pi \mathbf{a})}$ can also be used to obtain **pz**. If $\mathbf{i} = \sqrt{-1}$ and $\mathbf{z} = \mathbf{x + iy}$, the Westergaard stress function **Z(z)** is given by:

$$\mathbf{Z(z)} = \mathbf{z}\sigma_n/\sqrt{(\mathbf{z}^2 - \mathbf{a}^2)} \Rightarrow \mathbf{Z'(z)} = \mathbf{dZ/dz} = -\mathbf{a}^2\sigma_n/(\mathbf{z}^2 - \mathbf{a}^2)^{3/2} \tag{10}$$

This function can be used to analytically solve the linear elastic stress analysis problem when the cracked infinite plate is loaded by a uniform bi-axial nominal stress state whose contour conditions are given by $\sigma_x(\mathbf{z} \to \infty) = \sigma_y(\mathbf{z} \to \infty) = \sigma_n$. But it can be easily adapted to also obtain the stresses in the uniaxially tensioned plate, whose contour conditions are $\sigma_x(\mathbf{z} \to \infty) = \mathbf{0}$ and $\sigma_y(\mathbf{z} \to \infty) = \sigma_n$, simply by [3]:

$$\begin{cases} \sigma_x = \mathbf{Re(Z)} - \mathbf{yIm(Z')} - \sigma_n \\ \sigma_x = \mathbf{Re(Z)} + \mathbf{yIm(Z')} \\ \tau_{xy} = -\mathbf{yRe(Z')} \end{cases} \tag{11}$$

A constant term $-\sigma_n$ has been summed to σ_x in order to obey the contour condition $\sigma_x(\infty) = \mathbf{0}$ in this uniaxial case, in the same way that it can be done when the stresses are generated from the Williams' series. This simple trick can be used to solve the uniaxial problem because a constant σ_x does not affect the stress field near the crack tip or, in other words, because the crack concentration effect does not affect the stresses parallel to its (**x**) direction.
To calculate and to visualize the **pz** elastic-plastic frontier, it is worth to rewrite **Z** and **Z'** in polar coordinates centered at the crack tip:

$$\begin{cases} \mathbf{Z} = \dfrac{[\mathbf{a} + (\mathbf{r}\cdot\cos\theta) + \mathbf{i}(\mathbf{r}\cdot\sin\theta)]\cdot\sigma_n}{\sqrt{[\mathbf{a} + (\mathbf{r}\cdot\cos\theta) + \mathbf{i}(\mathbf{r}\cdot\sin\theta)]^2 - \mathbf{a}^2}} \\ \mathbf{Z}' = \dfrac{-\mathbf{a}^2\cdot\sigma_n}{\left\{[\mathbf{a} + (\mathbf{r}\cdot\cos\theta) + \mathbf{i}(\mathbf{r}\cdot\sin\theta)]^2 - \mathbf{a}^2\right\}^{3/2}} \end{cases} \tag{12}$$

It is now possible to obtain an equation to describe the **pz** elastic-plastic frontier around the crack tip by Mises in plane stress: just substitute (12) into (11), crank the algebra to obtain σ_x, σ_y and τ_{xy}, superpose these values by Mises and equate the result to the yield strength:

$$\begin{aligned} \Bigg\{ & \left[\mathbf{Re}\left(\frac{(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)\cdot\sigma_n}{\sqrt{(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)^2 - \mathbf{a}^2}}\right) - \mathbf{yIm}\left(\frac{-\mathbf{a}^2\cdot\sigma_n}{[(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)^2 - \mathbf{a}^2]^{3/2}}\right) - \sigma_n\right]^2 + \\ & + \left[\mathbf{Re}\left(\frac{(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)\cdot\sigma_n}{\sqrt{(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)^2 - \mathbf{a}^2}}\right) + \mathbf{yIm}\left(\frac{-\mathbf{a}^2\cdot\sigma_n}{[(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)^2 - \mathbf{a}^2]^{3/2}}\right)\right]^2 - \\ & - \left[\mathbf{Re}\left(\frac{(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)\cdot\sigma_n}{\sqrt{(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)^2 - \mathbf{a}^2}}\right) - \mathbf{yIm}\left(\frac{-\mathbf{a}^2\cdot\sigma_n}{[(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)^2 - \mathbf{a}^2]^{3/2}}\right) - \sigma_n\right]\cdot \\ & \cdot\left[\mathbf{Re}\left(\frac{(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)\cdot\sigma_n}{\sqrt{(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)^2 - \mathbf{a}^2}}\right) + \mathbf{yIm}\left(\frac{-\mathbf{a}^2\cdot\sigma_n}{[(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)^2 - \mathbf{a}^2]^{3/2}}\right)\right] + \\ & + \mathbf{3}\cdot\left[-\mathbf{yRe}\left(\frac{-\mathbf{a}^2\cdot\sigma_n}{[(\mathbf{a} + \mathbf{r}\cdot\cos\theta + \mathbf{i}\cdot\mathbf{r}\sin\theta)^2 - \mathbf{a}^2]^{3/2}}\right)\right]^2\Bigg\}^{1/2} = \mathbf{S_Y} \end{aligned} \tag{13}$$

This equation can be solved by numerical methods using the same techniques discussed above: for each θ value, the corresponding **r** is calculated to obey (13), localizing in this way the desired **pz** frontier. This same process can be easily repeated to obtain the so-called Westergaard **pz** in plane strain. Some results of these long calculations are shown in Fig.3.

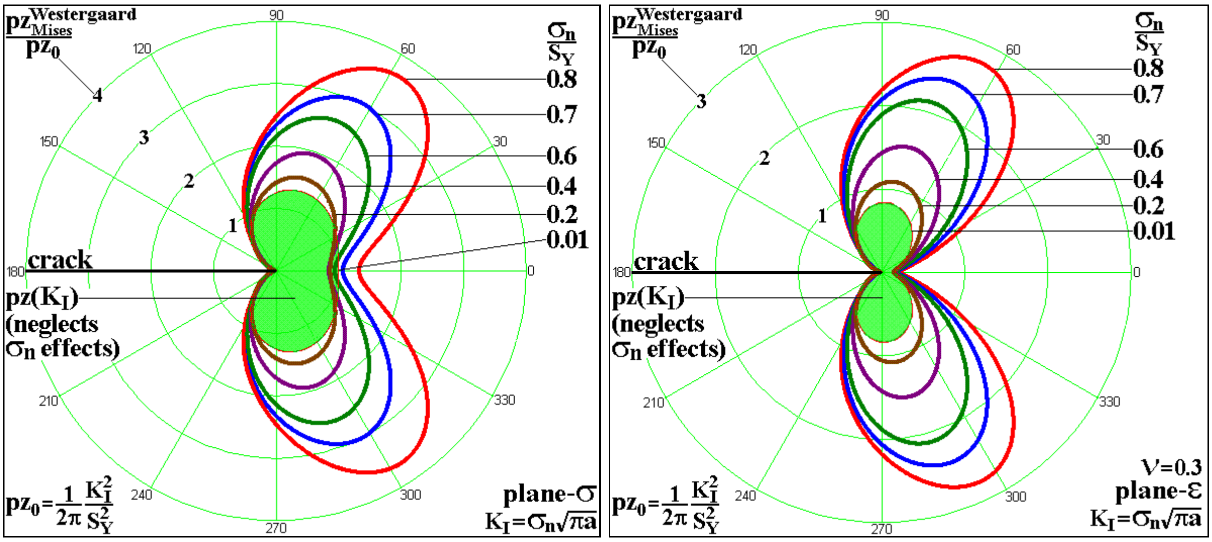

Fig.3: Plastic zones ahead a crack tip in an infinite plate calculated using Mises yielding criterion and the stresses generated by the complete Westergaard stress function, when it is tensioned by a nominal stress σ_n perpendicular to the crack, in plane stress and in plane strain.

COMPARING THE PLASTIC ZONES ESTIMATED BY INGLIS AND BY WESTERGAARD

Fig.4 compares **pz**s calculated by Inglis (assuming that the crack is an elliptical notch of tip radius ρ = **CTOD/2**), and by Westergaard (but without using the simplification expressed in (2) required to generate the classical $\mathbf{K_I} = \sigma\sqrt{(\pi a)}$ Inglis solution). As these curves were generated from completely different equations, their near coincidence is certainly *not* fortuitous. Therefore, the large σ_n effect on the size and the shape of the plastic zone predicted by these rigorous solutions should not be neglected in practice.

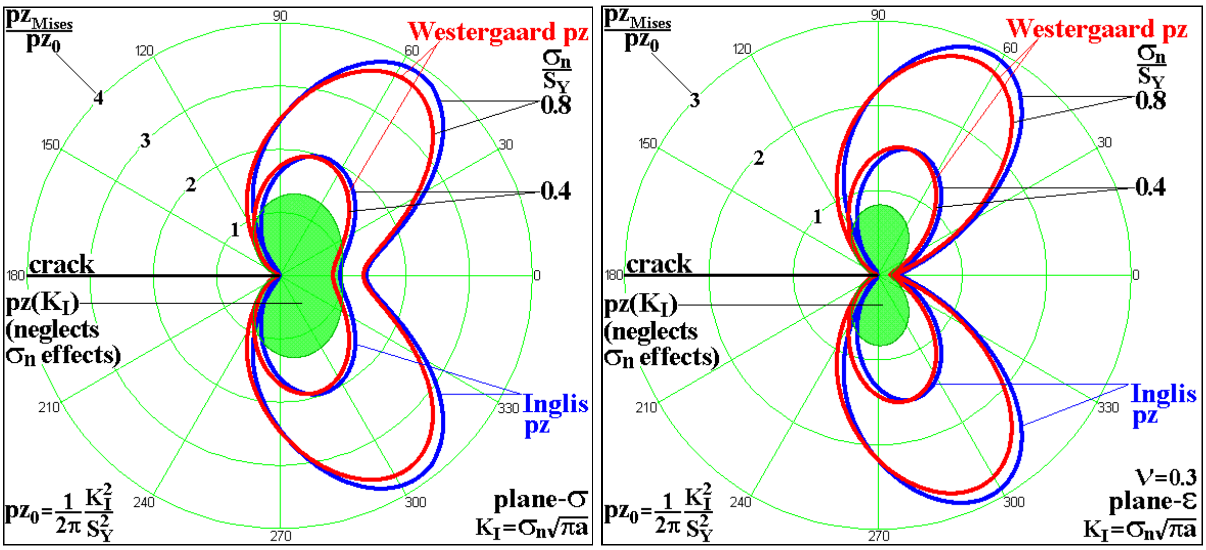

Fig.4: Inglis with ρ = **CTOD/2** and Westergaard **pz**s under plane stress and plane strain.

This point must be emphasized. It is convenient to assume that the plastic zone depends only on $\mathbf{K_I}$, neglecting thus the σ_n effect, but it is the **pz** size that *validates* LEFM predictions.

Therefore, underestimated **pz**s can be a practical problem, since nominal stresses as high as **0.8·S$_Y$** are common in real structures. Under such a high load, the σ_n effect on **pz** can be, and in general it probably is, very significant. Moreover, the nominal stress σ_n depends also on the structure shape, a possible reason for some specimen shape problems in fracture and fatigue testing.
To finalize this short note, it is interesting to point out that: (i) the Inglis and Westergaard **pz**s can visually coincide with each other simply by using **b** = **CTOD/2** instead of ρ = **CTOD/2** in the elliptical notch crack model, see Fig.5; and (ii) still better **pz** estimates can be obtained applying the Inglis equilibrium trick to correct the estimates presented here, see [4].

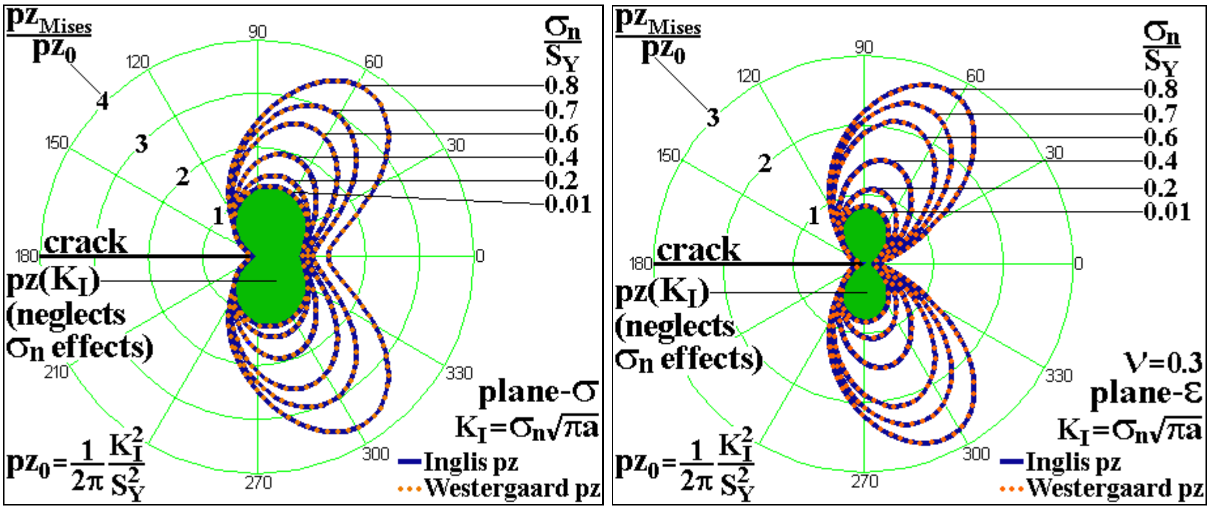

Fig.5: Inglis with **b** = **CTOD/2** and Westergaard **pz**s in plane stress and in plane strain.

CONCLUSIONS

The nominal stress σ_n significantly affects the size and the shape of the plastic zones ahead of crack tips, as illustrated by the rigorous solution of the Irwin crack problem. Therefore, contrary to what is usually accepted and taught in the traditional LEFM literature, the plastic zones do ***not*** depend only on the magnitude of the stress intensity factor **K$_I$**. This fact has important consequences, as it can be used to seriously question the similitude principle, one milestone in the practice of mechanical design against fracture. Thus, it should be better explored and understood.

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