ON INTEGRAL AND ENCLOSING SURFACE APPROACHES TO CALCULATE EQUIVALENT RANGES IN NON-PROPORTIONAL STRESS OR STRAIN HISTORIES

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Abstract. *Non-proportional (NP) multiaxial fatigue damage occurs when the principal stress directions vary during the loading induced by several independent sources, such as out-of-phase bending and torsion moments. A critical issue in multiaxial damage calculation in NP histories is to find the equivalent stress or strain ranges and mean components associated with each rainflow-counted cycle of the stress or strain path. A traditional way to find such ranges is to use enclosing surface methods, which search for convex enclosures, such as balls or prisms, of the entire history path in stress or strain diagrams. This work deals with how to compute equivalent stress or strain ranges in multiaxial NP histories using enclosing surface methods. All existing enclosing surface methods are presented and compared using results from more than* 3×10^6 *Monte Carlo simulations of random and especially chosen path topologies in two to five-dimensional stress or strain diagrams. A new approach to evaluate equivalent stress and strain ranges in NP histories is also presented, called the Moment Of Inertia (MOI) method. The MOI method is not based on enclosing surfaces, it is an integral approach which assumes instead that the path contour in the stress or strain diagram is a homogeneous wire with a unit mass. The center of mass of such wire gives then the mean component of the path, while the moments of inertia of the wire can be used to obtain the equivalent stress or strain ranges.*

Keywords: *multiaxial fatigue, equivalent stress, enclosing surfaces*

1. INTRODUCTION

For multiaxial variable amplitude (VA) loadings, in special for non-proportional (NP) load histories, it is usually not evident how to define and identify stress or strain ranges. The loading path, represented e.g. in a Mises diagram, can in such cases have a generic curved shape spanning infinitely many stress states, without clear peaks or valleys.

Consider a periodic load history that repeatedly follows a given loading path domain $\boldsymbol{D}$.. The larger fatigue damage associated with this path depends on the maximum shear stress range induced by it at the critical point, $\Delta\tau_{max}$. However, it is not easy to define the $\Delta\tau_{max}$ acting on a given plane at the critical point, since a generic NP loading path $\boldsymbol{D}$ results in NP variations of the shear stresses τ_B and τ_{B2} that act parallel to the critical plane, and both τ_B and τ_{B2} influence the growth of shear cracks along it. To calculate the maximum strain range $\Delta\tau_{max}$ at the critical plane, it is necessary to draw the path $\boldsymbol{D}$ (of the periodic stress history) along a $\tau_B \times \tau_{B2}$ diagram, as shown in Fig. 1.

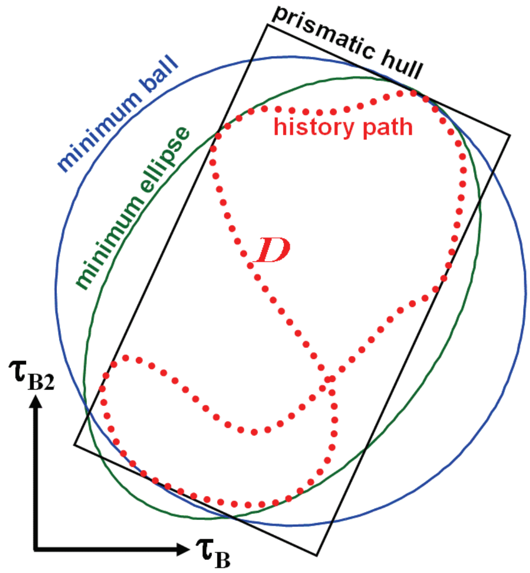

Figure 1: Stress history path $\boldsymbol{D}$ in the $\tau_B \times \tau_{B2}$ diagram, enclosed in surfaces based on circles (balls), ellipses and rectangular prisms.

For a complex-shaped history such as the one shown in the figure, it is not easy to decide how to obtain the effective $\Delta\tau_{max}$. The so-called enclosing surface methods try to find circles, ellipses or rectangles that contain the entire path (in the 2D case). In a nutshell, in the 2D case, the Minimum Ball (MB) method (Dang Van, 1999) searches for the circle with minimum radius that contains $\boldsymbol{D}$; the minimum ellipse methods (Freitas *et al.*, 2000; Gonçalves *et al.*, 2005; Zouain *et al.*, 2006) search for an ellipse with semi-axes a and b that contains $\boldsymbol{D}$ with minimum area $\pi{\cdot}a{\cdot}b$ or minimum norm $(a^2 + b^2)^{1/2}$; and the maximum prismatic hull methods (Gonçalves *et al.*, 2005; Mamiya *et al.*, 2009) search among the smallest rectangles that contain $\boldsymbol{D}$ the one with maximum area or maximum diagonal (this is a max-min search problem). The value of $\Delta\tau_{max}$ in Fig. 1 would either be assumed as the value of the circle diameter, or twice the ellipse norm, or the rectangle diagonal.

Such enclosing surface methods can be extended to histories involving more than two stress components. E.g., if the history path is plotted in a 3D diagram representing 3 stress components, the enclosing surface methods will search for spheres, ellipsoids or rectangular prisms. For higher dimension diagrams, the search is for hyperspheres, hyperellipsoids, and rectangular hyperprisms. However, this practice can lead to significant errors, since each enclosing surface will reflect an effective range calculated on different planes at different points in time. The recommended approach for general 6D histories involving all stress components is then to project them onto candidate planes, resulting in each case in searches for effective ranges in 2D diagrams $\tau_B \times \tau_{B2}$.

The enclosing surface methods are briefly described in the following sections. Their framework is based on deviatoric stress diagrams and Mises stress parameters.

2. MINIMUM BALL METHOD

When dealing with multiaxial fatigue problems, Dang Van (1999) realized that the search for an effective stress range must take place on the deviatoric stress space. For periodic elastic histories, the mesoscopic stresses and strains in the critically oriented grain should stabilize by the process of elastic shakedown, generating a local residual stress $[\sigma_{ij}]_{res}$ at such critical grain. Dang Van assumed that the subsequent mesoscopic (μ) stress history at such grain, after the stabilization, is related to the macroscopic (M) history through

$$[\,\sigma_{ij}(t)\,]_{\mu} = [\,\sigma_{ij}(t)\,]_{M} + dev[\,\sigma_{ij}\,]_{res} \tag{1}$$

where $dev[\sigma_{ij}]_{res}$ is the deviatoric part of the residual stresses tensor stabilized in that grain.

The calculation of the mesoscopic stresses in Dang Van's model can be interpreted as a hardening problem, caused by elastic shakedown. When the periodic macroscopic history is represented in the deviatoric space, Dang Van assumes that the stabilized residual stress is the vector from the center of the minimum ball that circumscribes the history to the origin of the diagram. The word "ball" is used here to describe a circle, sphere or hypersphere, respectively for 2D, 3D or higher dimension stress histories.

Dang Van is a type of Minimum Ball (MB) method where each stress state along the history path is compared to a limiting stress level to predict infinite life. However, it is not useful to calculate finite fatigue lives, since it does not deal with stress (or strain) ranges, only with individual stress states.

But the same MB circumscribed to the macroscopic history can be used to estimate an effective Mises stress range $\Delta\sigma_{Mises}$. The diameter d of such MB, if represented in Papadopoulos' 5D deviatoric stress sub-space (Papadopoulos *et al.*, 1997), is equal to $\Delta\sigma_{Mises}$. Therefore, the effective shear range $\Delta\tau_{max}$, Mises range $\Delta\sigma_{Mises}$, and octahedral shear range $\Delta\tau_{Mises}$ can all be estimated from d using the MB method by

$$\Delta\sigma_{Mises} = 3\cdot\Delta\tau_{Mises}/\sqrt{2} = \Delta\tau_{max}\sqrt{3} = (2\tau_a)\cdot\sqrt{3} = |\Delta\bar{S}'| = d \equiv L\cdot\lambda_{MB} \tag{2}$$

where L is the longest chord in the history (or, more formally, the maximum Euclidean distance in the transformed space between any two points along the history path, measured in stress units) and λ_{MB} is a dimensionless parameter defined as the ratio between the Mises stress range and L.

In the 2D case, if any two points from the history define the diameter of a circle that contains the entire path, then their distance L is equal to the diameter d, therefore $\lambda_{MB} = 1.0$. A notable 2D case is for a path forming an equilateral triangle, where $\lambda_{MB} = 2/\sqrt{3} \cong 1.155$. For any other 2D path, it is found that $1.0 \le \lambda_{MB} \le 1.155$.

3. MINIMUM ELLIPSOID METHODS

The Minimum Ball (MB) method is not efficient to represent the behavior of NP histories. For instance, it would predict the same Mises ranges for a NP 90° out-of-phase circular path and a proportional path defined by a diameter of this circle, both resulting in $\lambda_{MB} = 1.0$. But a higher value of λ_{MB} would certainly be expected for the NP history.

To solve this problem, Freitas *et al.* (2000) proposed the Minimum Circumscribed Ellipsoid (MCE) method. It searches for an ellipse (or ellipsoid or hyperellipsoid, for higher dimensions) that circumscribes the entire stress history, with its longest semi-axis a_1 equal to the radius of the minimum ball, and with the smallest possible values for the remaining semi-axes a_i ($i > 1$). The Mises ranges are defined by

$$\Delta\sigma_{Mises} = 2 \cdot \sqrt{\sum_{i=1}^{dim} a_i^2} \equiv 2 \cdot F \tag{3}$$

where dim is the dimension of the history path, $2 \leq dim \leq 5$, and F is defined as the Frobenius norm of the ellipsoid, which is equal to the square root of the sum of the squares of the ellipsoid semi-axes. Here, the Frobenius norm is essentially an Euclidean distance (or Euclidean norm) between the origin and a point with coordinates $(a_1, a_2, ..., a_{dim})$, since the axes of the reduced stress space are orthonormal. In the case of tensors, the Euclidean norm is commonly called the Frobenius norm, usually abbreviated as F-norm.

The ratio λ_{MCE} between the Mises ranges calculated by the MCE method and the longest chord L reproduces experimental data better than λ_{MB} generated by the MB method. In the 2D case, a NP circular path would result in a ratio $\lambda_{MCE} = \sqrt{2}$ instead of the proportional value 1.0, which is much more reasonable than the Minimum Ball prediction. It is also found that any 2D path results in $1.0 \leq \lambda_{MCE} \leq \sqrt{2}$, with the maximum value occurring e.g. for circular and square paths.

The downside of the MCE method is the requirement that the longest semi-axis must be equal to the radius of the Minimum Ball. For a rectangular path, this requirement results in a circle as the minimum circumscribed ellipse, with $\lambda_{MCE} = \sqrt{2} \cong 1.414$. But this would be true even for very elongated rectangles with very low aspect ratios between their side lengths. The MCE would thus predict $\lambda_{MCE} = \sqrt{2}$ for an almost proportional rectangular path, instead of the expected value of 1.0.

A possible alternative to the MCE method is to search for the Minimum Volume Ellipsoid (MVE), also known as the Löwner-John Ellipsoid. In the 2D case, it is basically the search for an enclosing ellipse with minimum area. Such MVE method solves the issue with rectangular paths, however it tends to find ellipses with lower aspect ratios than expected. In addition, the search for such ellipsoid or hyperellipsoid can be computationally intensive for 3D or higher dimension histories.

Another alternative to the MCE method is the search for the Minimum F-norm Ellipsoid (MFE) (Gonçalves *et al.*, 2005). Instead of searching for the minimum volume (or area), the MFE looks for the ellipse, ellipsoid, or hyperellipsoid with minimum value of its F-norm F, defined in Eq. (3). Zouain *et al.* (2006) present an efficient (although computationally intensive) method to numerically find such MFE.

The ratios between the Mises stress or strain ranges $2 \cdot F$, calculated from the MCE, MVE and MFE methods, and the longest chord L are defined, respectively, as λ_{MCE}, λ_{MVE} and λ_{MFE}. All these ratios must be greater than or equal to 1.0. In the 2D case, a notable path is the one with the shape of an equilateral triangle with sides L (which are also its longest chords), where the resulting hull is a circle with diameter $d = 2L/\sqrt{3}$ and F-norm $F = d\sqrt{2}$, resulting in $\lambda_{MCE} = \lambda_{MVE} = \lambda_{MFE} = 2 \cdot F/L = 2\sqrt{2}/\sqrt{3} \cong 1.633$. For any other 2D path, it is found that $1.0 \leq \lambda_{MCE} \leq 1.633$ and $1.0 \leq \lambda_{MFE} \leq 1.633$, however λ_{MVE} can reach values beyond 2.0 when a very elongated enclosing ellipse is the solution with minimum area, an indication that the MVE method can be very conservative.

4. MAXIMUM PRISMATIC HULL METHODS

Another class of enclosing surface methods tries to find a rectangular prism with sides $2a_1, ..., 2a_{dim}$ that encloses a load history path, where dim is the dimension of the considered space. There are essentially 2 methods to fit rectangular prisms to the history path.

The first is the Maximum Prismatic Hull (MPH). This method searches for the smallest rectangular prism that encloses the history (the minimum prism), for each possible orientation of the prism. Among them, the one with highest F-norm is chosen. The F-norm and resulting Mises ranges are the same defined in Eq. (3), except that here a_i are the semi-lengths (half the length) of the sides of the rectangular prism. The MPH was originally proposed by Gonçalves *et al.* (2005) for sinusoidal time histories, and later extended by Mamiya *et al.* (2009) for a general NP loading.

Another prismatic hull method is the Maximum Volume Prismatic Hull (MVPH), which searches among the minimum prisms the one with maximum volume. Although the search is for a maximum volume, the F-norm is also used to compute the Mises range. In the 2D case, the MVPH method is essentially the search, among the minimum rectangles that enclose the entire path, of the one with maximum area (it's a max-min problem).

The ratios between the Mises stress or strain ranges $2 \cdot F$, calculated from the MPH and MVPH methods, and the longest chord L are defined, respectively, as λ_{MPH} and λ_{MVPH}. These ratios are, in average, very close to each other. For a history path with dimension dim, it is found that $1 \leq \lambda_{MVPH} \leq \lambda_{MPH} \leq \sqrt{dim}$.

5. THE MOMENT OF INERTIA (MOI) METHOD

An improved equivalent stress method, called the Moment Of Inertia (MOI) method, is proposed here. Instead of using an enclosing surface, it is based on an integral approach that takes into account the shape of the load path. It is useful to calculate alternate and mean components of complex NP load histories, which must first be represented in a 2D stress-subspace of the transformed 5D Euclidean stress-space proposed by Papadopoulos *et al.* (1997).

Similarly to the enclosing surface methods, the MOI should only be applied to 2D histories, involving one normal and one shear stress or strain components (e.g. represented in the $\sigma_x \times \tau_{xy}\sqrt{3}$ diagram) or two shear components acting on the same plane (e.g. represented in the $\tau_{xz}\sqrt{3} \times \tau_{xy}\sqrt{3}$ diagram). It would lead to significant errors if directly applied to 3D, 4D or 5D load histories, because the MOI would be calculated on different planes at different instants in time (Socie, 1999). Instead, any 3D, 4D or 5D history should first be projected onto a candidate plane. Then, the history of the two shear stresses (or strains) acting parallel to the crack plane would be represented in a 2D diagram, where the MOI method would be applied. Thus, only the 2D formulation of the MOI method will be presented here.

The MOI method assumes that the 2D path/domain $\mathcal{D}$ of the stress history, represented by a series of points (X, Y) containing the stress or strain variations along it, is a homogeneous wire with unit mass. Note that X and Y can have stress or strain units, but they are completely unrelated to the directions x and y usually associated with the material surface. The mean component of $\mathcal{D}$ is assumed, in the MOI method, to be located at the center of gravity of this imaginary homogeneous wire shaped as the history path. Such center of gravity is located at the perimeter centroid (X_c, Y_c) of $\mathcal{D}$, calculated from contour integrals along the entire path

$$X_c = \frac{1}{p} \cdot \oint X \cdot dp, \quad Y_c = \frac{1}{p} \cdot \oint Y \cdot dp, \quad p = \oint dp \tag{4}$$

where dp is the length of an infinitesimal arc of the path and p is the path perimeter, see Fig. 2.

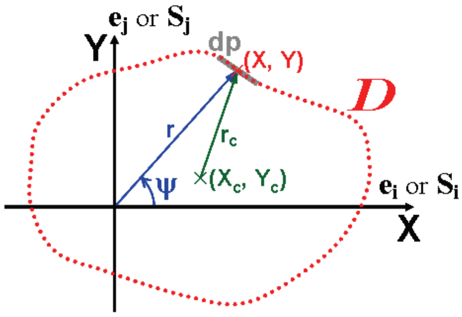

Figure 2: Load history path, assumed as a homogeneous wire with unit mass.

Note that this perimeter centroid (PC) is in general different from the area centroid (AC), which is the center of gravity of a uniform density sheet bounded by the shape of a closed path $\mathcal{D}$. The PC gives a much more reasonable estimate of the mean stress component.

The MOI method is so called because it makes use of the mass moments of inertia (MOI) of the homogeneous wire that is supposed analogous to the load path. These moments are first calculated with respect to the origin O of the diagram, assuming the wire has unit mass, resulting in

$$I_{XX}^{O} = \frac{1}{p} \cdot \oint Y^2 \cdot dp, \quad I_{YY}^{O} = \frac{1}{p} \cdot \oint X^2 \cdot dp, \quad I_{XY}^{O} = -\frac{1}{p} \cdot \oint X \cdot Y \cdot dp \tag{5}$$

Then, the moments of inertia of such unit mass wire, with respect to its center of gravity (X_c, Y_c), are obtained. They are computed from the moments of the path $\mathcal{D}$ domain with respect to its perimeter centroid (X_c, Y_c), which are easily obtained from the parallel axis theorem, assuming a unit mass:

$$I_{XX} = I_{XX}^{O} - Y_c^{\,2}, \quad I_{YY} = I_{YY}^{O} - X_c^{\,2}, \quad I_{XY} = I_{XY}^{O} + X_c \cdot Y_c \tag{6}$$

The MOI method assumes that the deviatoric stress (or strain) ranges, $\Delta S \equiv \Delta\sigma_{Mises}$ (or $\Delta e \equiv \Delta\varepsilon_{Mises}$), depend on the mass moment of inertia I_{ZZ} with respect to the perimeter centroid, perpendicular to the X-Y plane. This is physically sound, since history paths further away from the PC will contribute more to the effective range and amplitude, which is coherent with the fact that wire segments further away from the center of gravity of an imaginary homogeneous wire contribute more to its MOI. From the perpendicular axis theorem, which states that $I_{ZZ} = I_{XX} + I_{YY}$, and from a dimensional analysis, it is found that

$$\frac{\Delta\sigma_{Mises}}{2} \text{ or } \frac{\Delta\varepsilon_{Mises}}{2} = \sqrt{3 \cdot I_{ZZ}} = \sqrt{3 \cdot (I_{XX} + I_{YY})} \tag{7}$$

The factor $\sqrt{3}$ is introduced to guarantee that a proportional loading path, represented by a straight segment with length L and unit mass $m = 1$, will result in the expected range $\Delta\sigma_{Mises}$ or $\Delta\varepsilon_{Mises}$ equal to L. Note that the above definitions are coherent, since they are independent of the orientation of the X-Y system because $I_{XX} + I_{YY}$ is an invariant, equal to the sum of the principal MOI $I_1 + I_2$ of the homogeneous wire.

In the next section, all enclosing surface methods presented in this paper are evaluated and compared.

6. COMPARISON AMONG THE ENCLOSING SURFACE METHODS

Figure 3 shows the enclosing surfaces obtained from all presented methods for a rectangular history path in a reduced 2D sub-space, and their ratios λ between the Mises ranges and longest chord L. Note that, in this example, L is the diagonal of the rectangular path.

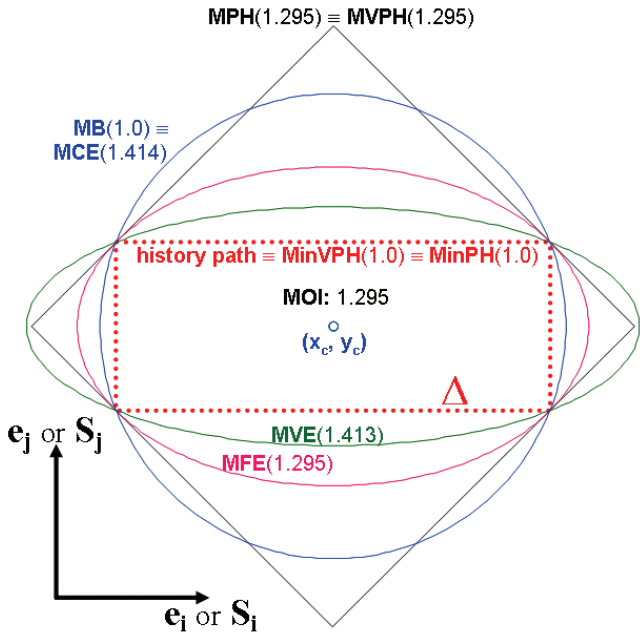

Figure 3: Values of the λ Mises stress (or strain) range ratio for the MB, MCE, MVE, MFE, MPH, MVPH, MinPH, MinVPH and MOI methods for a rectangular 2D history path, where Min stands for minimum (volume or F-norm).

Experimental results suggest that the expected ratio λ in this example is about 1.3. However, the MB method predicts $\lambda_{MB} = 1.0$, which is very non-conservative. The MB assumes that such rectangular path would have the same Mises range L as a straight path along one of its diagonals, which is not reasonable. The MCE method, on the other hand, overestimates λ, obtaining $\lambda_{MCE} = \sqrt{2} \cong 1.414$. The MCE method finds the same circle from the MB to enclose such history, even though the aspect ratio of this rectangular path is very different from 1.0, suggesting instead the use of an elongated elliptic hull.

The MVE method also tends to overestimate λ, obtaining in this example $\lambda_{MVE} = 1.413$. In the search for the minimum area (or volume, for higher dimension diagrams), the MVE method ends up finding overly elongated ellipses ($b << a$), which have a small area $\pi \cdot a \cdot b$ due to the very low value of b but an unrealistically high F-norm $(a^2+b^2)^{1/2}$ due to the high value obtained for a. Thus, λ_{MVE} overestimates the ratio λ, since it is calculated from this unrealistic F-norm, and not from the area (or volume).

Among the ellipsoid hull methods, the MFE gives the best predictions, resulting in $\lambda_{MFE} = 1.295$, with an enclosing ellipse with a much more reasonable aspect ratio than the ones from the MCE and MVE methods, see Fig. 3.

Both MPH and MVPH methods obtain in this example $\lambda_{MPH} = \lambda_{MVPH} = 1.295$, which exactly agrees with the MFE prediction. Note however that the MPH and MFE methods are not equivalent, since they result in slightly different λ values between them for other history paths, as shown by Castro *et al.* (2009).

Figure 3 also shows the prismatic hulls MinPH and MinVPH with minimum (instead of maximum) F-norm and volume (or area, in 2D), respectively. In this example, these rectangular hulls would coincide with the original rectangular path, wrongfully predicting $\lambda = 1$. This counter-example shows why no prismatic hull method with minimum F-norm or volume has been proposed.

In summary, the MB method tends to underestimate the Mises stress or strain range ratio λ, while the MCE and MVE overestimate it. The MFE, MPH and MVPH give very similar (although, in general, different) predictions.

But the above considerations are based on a single example. To really compare all enclosing surface methods, it is necessary to study all possible history path topologies in 2D, 3D, 4D and 5D deviatoric stress or strain spaces. Monte Carlo simulations are performed for $3 \cdot 10^6$ random 2D history paths, in addition to a few selected paths to try to cover all possible path topologies. All enclosing surface methods are applied to each of these simulated paths, to evaluate and compare the λ predictions. The following discussions focus on 2D paths, however similar conclusions are found for 3D, 4D and 5D histories.

The simulations show that the MPH and MVPH methods have a very good agreement, except for low values of λ. It is found that $\lambda_{MVPH} \leq \lambda_{MPH}$ and, in average, λ_{MVPH} is about *98.6%* of λ_{MPH}, with a standard deviation of only *1.8%*.

The MPH and MFE methods are coherent, however they can lead to very different λ predictions. It is found that $\lambda_{MFE} \geq \lambda_{MPH}$ and, in average, λ_{MPH} is about *92.9%* of λ_{MFE}, with a standard deviation of *4.3%*.

The MVE method can severely (and wrongfully) overestimate λ, in special for low values of λ_{MPH}, associated with almost proportional paths. As discussed before, almost proportional paths can lead to overly elongated ellipses in the MVE method, which can have a small area but an unrealistically large F-norm, leading to λ_{MVE} values larger than 2.0 in some extreme cases.

Also, it is found that λ_{MCE} overestimates λ, in special for low values of λ_{MFE}, associated with almost proportional paths. For instance, for an almost proportional history defined by a rectangular path with very low aspect ratio, the expected λ would be close to *1.0* (which is the expected value of λ for proportional histories), however the MCE method would circumscribe a circle (instead of an elongated ellipse) to such elongated rectangular path, wrongfully predicting $\lambda_{MCE} = \sqrt{2}$, revealing the inadequacy of the MCE method.

Finally, the MB method can severely (and wrongfully) underestimate λ, except for almost proportional load histories (where $\lambda_{MB} \cong \lambda_{MFE} \cong 1.0$).

7. CONCLUSIONS

In this work, all enclosing surface methods from the literature were reviewed and compared, and a new integral method was proposed. The conclusions from the comparisons are:

1. the only recommended ellipsoid hull is the Minimum F-norm Ellipsoid (MFE), which results in similar (but not equal) λ predictions when compared to the prismatic hull methods;
2. the Minimum Circumscribed Ellipsoid (MCE), Minimum Volume Ellipsoid (MVE), and Minimum Ball (MB) methods may result in very poor predictions of the stress or strain amplitudes;
3. the Minimum F-norm Ellipsoid and all Maximum Prismatic Hull (MPH) models are efficient to predict equivalent amplitudes in NP histories, however they do not perform well in non-convex paths such as cross or star-shaped paths; and
4. the proposed MOI Method is an integral approach that is able to reproduce the results of the best enclosing surface methods, with a much simpler calculation algorithm that does not depend on the search of enclosing surfaces, working well even in complex non-convex paths such as cross or star-shaped paths.

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