MULTI-SURFACE INCREMENTAL PLASTICITY FORMULATION WITH NON-LINEAR KINEMATIC MODELS FOR NON-PROPORTIONAL VARIABLE AMPLITUDE HISTORIES

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ABSTRACT

In this work, an efficient incremental plasticity framework is proposed to obtain the multiaxial stress-strain history in the non-proportional (NP) case under complex variable-amplitude (VA) loadings, considering both isotropic and kinematic hardening. The algorithm framework is entirely developed in Papadopoulos' 5D deviatoric sub-space, which is more computationally efficient than Ilouchine's 9D or Voigt's generalized 6D representations, being able to deal with very long load histories. The flow rules and kinematic translation rules have to be adapted to this 5D notation (which becomes 3D in the plane stress case). The algorithm is able to reproduce the Mròz multi-surface model and all non-linear kinematic (NLK) models proposed in the literature using the same notation and variables, therefore providing a unified framework to directly compare such models and even to mix their surface translation rules. For instance, Jiang-Sehitoglu's efficient translation rules could be used together with Mròz multi-surface model to deal with VA loadings without the drawbacks of some NLK models (which fail to reproduce histories with decreasing amplitude sections, which might be present in VA fatigue problems). A general class of translation rules is proposed, which can reproduce all hardening rules studied by Jiang and Sehitoglu, in addition to the Mròz and Garud rules, among others. It is found that the algorithm is able to reproduce the experimental hysteresis loops from NP VA loadings with a relatively low computational cost.

KEYWORDS

Incremental plasticity, multiaxial fatigue, multi-surface formulation, non-linear kinematic models, non-proportional loading, variable amplitude loading.

INTRODUCTION

In most engineering applications, either the stress or the strain history is known, but not both. When designing a new component, it is common to calculate or estimate the stress history from measured or specified design loads, whereas in most structural integrity evaluations (of an existing component) only the strain history can be measured using strain gage rosettes, for example. But the best multiaxial fatigue damage models require the knowledge of both the stress and the corresponding strain histories to quantify the consequent damage parameter.

For linear elastic histories, it is trivial to correlate the stresses with the strains using Hooke's law. But to properly reproduce the stress-strain hysteresis loops in NP elastoplastic histories, which depend on the load path, it is necessary to use incremental plasticity models to correlate infinitesimal changes in all stress components with the associated strain components, and vice-versa. They are based on 3 equations: the yield function, which describes combinations of stresses that lead to plastic flow; the flow rule, which describes the relationship between stresses and plastic strains; and the hardening rule, which defines how the yield criterion changes with plastic straining.

In this work, an incremental plasticity algorithm is proposed to predict the multiaxial stress-strain history of a material under complex variable-amplitude non-proportional loadings. The algorithm framework is developed in a computationally efficient 5D deviatoric sub-space $\bar{\mathbf{S}} \equiv \bar{\mathbf{S}}_{\mathbf{5D}}$, defined as

$$\bar{\mathbf{S}} \equiv \bar{\mathbf{S}}_{\mathbf{5D}} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1/2 & -1/2 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6}/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6}/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6}/2 \end{bmatrix}}_{\mathbf{A}} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy}\sqrt{2} \\ \tau_{xz}\sqrt{2} \\ \tau_{yz}\sqrt{2} \end{bmatrix} \equiv \mathbf{A} \cdot \bar{\sigma} \tag{1}$$

where A is a transformation matrix. This framework is able to combine multi-surface methods with non-linear kinematic models, reproducing effects such as dynamic recovery in variable amplitude histories.

KINEMATIC HARDENING MODELS

The Bauschinger effect, commonly called *kinematic hardening*, can be modeled in stress space, allowing the yield surface to translate with no change in its size or shape. So, in the deviatoric stress space, kinematic hardening maintains the radius $S_Y\sqrt{6}/3$ of the yield hypersphere fixed, while its center is translated, changing the generalized plastic modulus C, where S_Y is the material yield strength. Several models can be used to obtain the current value of C as the yield surface translates, to calculate the plastic strain increments. One of them is the multi-surface model, described next.
In the multi-surface model, several yield surfaces are considered, obtained from a discretization of the stress-strain curve. The first surface is the one that defines the elastic limit of the material, usually represented in the deviatoric space by a hypersphere with radius $r_1 = S_Y\sqrt{6}/3$, implying that the material is assumed purely elastic for plastic strains below 0.2%. Considering n_c surfaces, the values of $r_2, ..., r_{nc}$ are calculated from the cyclic effective stress-strain curve $\sigma_x\sqrt{6}/3 \times \varepsilon_{xp}$, where σ_x and ε_{xp} are obtained from uniaxial tests, see Fig. 1.

The multi-surface model assumes that the yield surfaces are rings with increasing radii r_i, initially concentric. The rings cannot intersect each other, except when they are tangent. While the current stress state, represented by the vector $\bar{\mathbf{S}}$, is moving in the deviatoric space inside the inner ring, with radius r_1, all stress and strain increments are purely elastic. When the vector $\bar{\mathbf{S}}$ touches the border of the inner ring, this ring becomes the *active surface* and starts translating until it touches the next one, termed the *target surface*, which has radius r_2. During this trajectory between r_1 and r_2, the plastic strain increment is calculated using the first plastic modulus $C = C_1$. The ring r_2 then becomes the active surface.

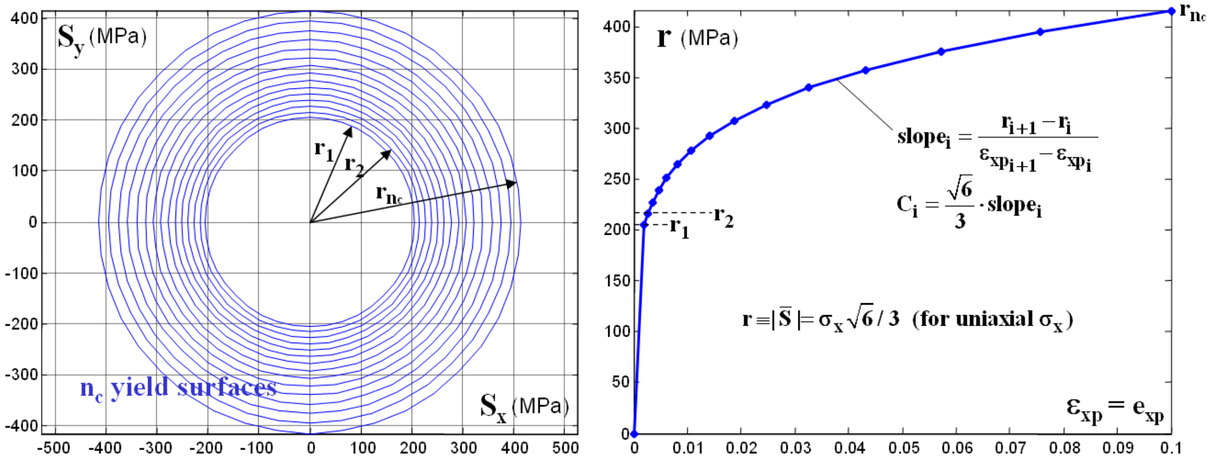

Fig. 1: Yield surfaces in the S_x-S_y deviatoric stress space, and correspondent radii obtained from the piecewise linearization of the cyclic effective stress-strain curve.

If the loading is further increased, both rings r_1 and r_2 are translated altogether as a rigid body, until touching the next *target surface* (ring) r_3. Analogously, during this trajectory between r_2 and r_3, the plastic strain increment is calculated using $C = C_2$. The ring r_3 becomes the active surface and the process continues, until some loading reversal makes the vector $\bar{\mathbf{S}}$ move inside the inner ring. During this trajectory inside the inner ring, the strain increments are purely elastic, no surface is active, and therefore none of the rings move. The rings will only move again when $\bar{\mathbf{S}}$ touches again the inner ring. Note however that the rings are not concentric anymore. The plastic memory of the material is stored in the model through these relative positions between the centers of the rings.

SURFACE TRANSLATION RULES

But, under non-proportional loading, the direction along which the deviatoric stress vector $\bar{\mathbf{S}}$ moves may be different from the ring translation direction. To calculate the ring translation direction in this general case, two rules have been proposed, one by Mròz [1] and another by Garud [2]. These models try to reproduce the multiaxial aspect of the Bauschinger effect, i.e., how the yielding in one direction can influence the kinematic hardening, not only in the opposite direction, but in all directions.

The Mròz rule assumes that the translation $\mathbf{d\bar{S}_{c_i}}$ of ring r_i occurs in a direction $\bar{\mathbf{v}}_\mathbf{M}$ parallel to the line that joins the current stress $\bar{\mathbf{S}}$ at ring r_i with the corresponding “image stress” $\bar{\mathbf{S}}_\mathbf{M}$ at the next ring r_{i+1}, which has the same normal unit vector $\bar{\mathbf{n}}_\mathbf{M}$, see Fig. 2. In this figure, $\bar{\mathbf{S}}_{\mathbf{c_i}}$ and $\bar{\mathbf{S}}_{\mathbf{c_{i+1}}}$ are the centers of rings r_i and r_{i+1} in the deviatoric space. The Mròz direction $\bar{\mathbf{v}}_\mathbf{M}$ is then obtained by

$$\left.\begin{aligned} \bar{\mathbf{S}}_\mathbf{M} &= \bar{\mathbf{S}}_{\mathbf{c_{i+1}}} + \bar{\mathbf{n}}_\mathbf{M} \cdot \mathbf{r_{i+1}} \\ \bar{\mathbf{S}} &= \bar{\mathbf{S}}_{\mathbf{c_i}} + \bar{\mathbf{n}}_\mathbf{M} \cdot \mathbf{r_i} \end{aligned}\right\} \Rightarrow \bar{\mathbf{v}}_\mathbf{M} = \bar{\mathbf{S}}_\mathbf{M} - \bar{\mathbf{S}} = (\bar{\mathbf{S}}_{\mathbf{c_{i+1}}} - \bar{\mathbf{S}}_{\mathbf{c_i}}) + \bar{\mathbf{n}}_\mathbf{M} \cdot (\mathbf{r_{i+1}} - \mathbf{r_i}) \qquad (2)$$

$$\bar{\mathbf{n}}_\mathbf{M} = \frac{\bar{\mathbf{S}} - \bar{\mathbf{S}}_{\mathbf{c_i}}}{|\bar{\mathbf{S}} - \bar{\mathbf{S}}_{\mathbf{c_i}}|} \qquad (3)$$

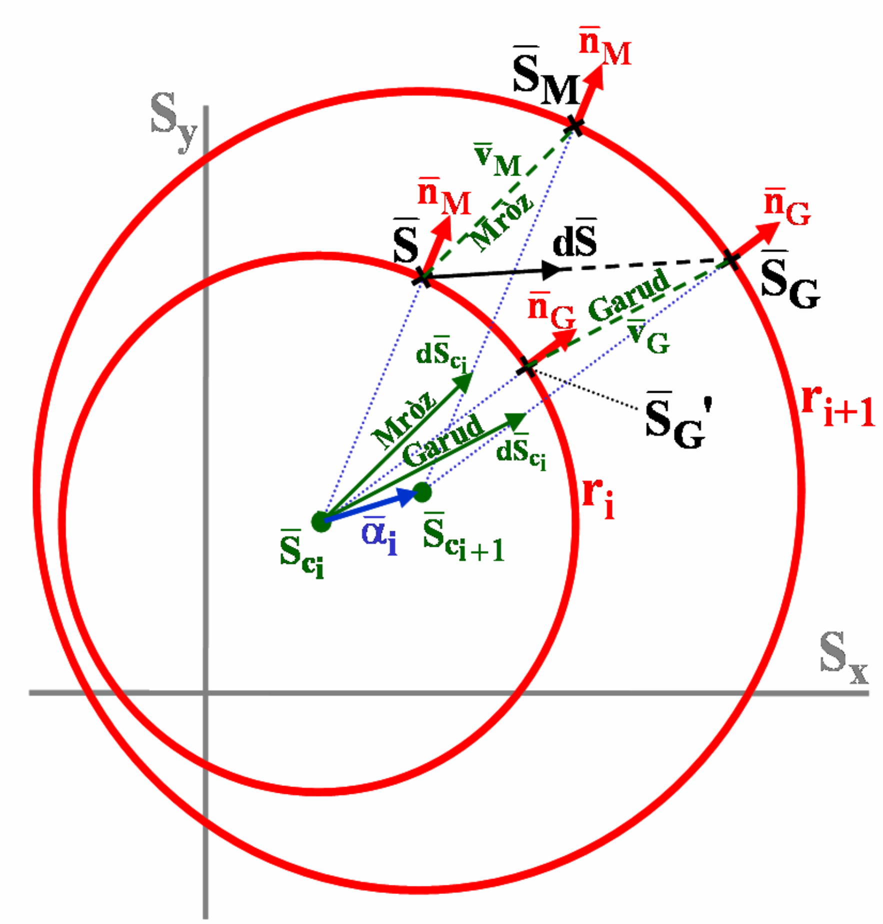

Fig. 2: Illustration of Mròz and Garud kinematic hardening rules.

However, the Mròz rule can induce a few numerical problems, which can result in rings intersecting in more than one point. Garud's rule does not have such numerical problems. It states that the translation of ring r_i is in a direction parallel to the line that joins the points $\bar{S}'_G$ and $\bar{S}_G$ shown in Fig. 2. The image stress $\bar{S}_G$ is commonly known as the *incremented stress state*, found from the intersection between ring r_{i+1} and the vector $\bar{S} + \alpha \cdot d\bar{S}$, where α ($\alpha > 0$) is a constant that can be geometrically determined from such intersection condition. The normal vector $\bar{n}_G$ of the incremented stress $\bar{S}_G$ is then

$$\bar{n}_G = \frac{\bar{S} - \bar{S}_{c_{i+1}} + \alpha \cdot d\bar{S}}{|\bar{S} - \bar{S}_{c_{i+1}} + \alpha \cdot d\bar{S}|} \tag{4}$$

The point $\bar{S}'_G$ is defined as the corresponding point of $\bar{S}_G$ at ring r_i with same normal unit vector $\bar{n}_G$, see Fig. 2. The translation of the center of ring r_i according to Garud is then parallel to the direction $\bar{v}_G$, where

$$\left.\begin{array}{r} \bar{S}_G = \bar{S}_{c_{i+1}} + \bar{n}_G \cdot r_{i+1} \\ \bar{S}_G{}' = \bar{S}_{c_i} + \bar{n}_G \cdot r_i \end{array}\right\} \Rightarrow \bar{v}_G = \bar{S}_G - \bar{S}_G{}' = (\bar{S}_{c_{i+1}} - \bar{S}_{c_i}) + \bar{n}_G \cdot (r_{i+1} - r_i) \tag{5}$$

In summary, while the Mròz rule tries to join the current point $\bar{S}$ from ring r_i and its corresponding point in ring r_{i+1}, the Garud rule tries to join the future point $\bar{S} + \alpha \cdot d\bar{S}$ from ring

r_{i+1} and its corresponding point in ring r_i. By dealing with a future point, the Garud rule guarantees that both rings will eventually become tangent at such future point, and anywhere else, as opposed to the Mròz model.

Mròz and Garud's multi-surface models can deal with complex variable amplitude (VA) histories, however their translation rules may fail to reproduce certain highly non-proportional (NP) out-of-phase loadings.

Non-Linear Kinematic (NLK) Hardening Models, on the other hand, are able to deal with complex dynamic recovery and radial return terms present in NP loadings, however they are not suitable to reproduce histories with decreasing amplitude sections, which might be present in VA fatigue problems.

But Chaboche's generalization [3] of the Armstrong-Frederick NLK rule [4] with multiple back-stresses can be interpreted as a multi-surface model, where the nested surfaces are connected by non-linear dampers instead of friction sliders [5]. Therefore, all multi-surface and NLK rules can be combined into the same framework, using generalized translation rules. In this work, the translation direction is then assumed to be represented by the vector

$$\bar{\mathbf{v}}_i = \bar{\mathbf{n}}\cdot(\mathbf{r}_{i+1} - \mathbf{r}_i) - \gamma(1-\delta)\cdot(\bar{\alpha}_i^{T}\cdot\bar{\mathbf{n}})\cdot\bar{\mathbf{n}} - \gamma\cdot\delta\cdot\bar{\alpha}_i \tag{6}$$

Note that this generalized surface translation direction includes radial return and dynamic recovery terms from the material calibration of the parameters γ and δ, see Fig. 3.

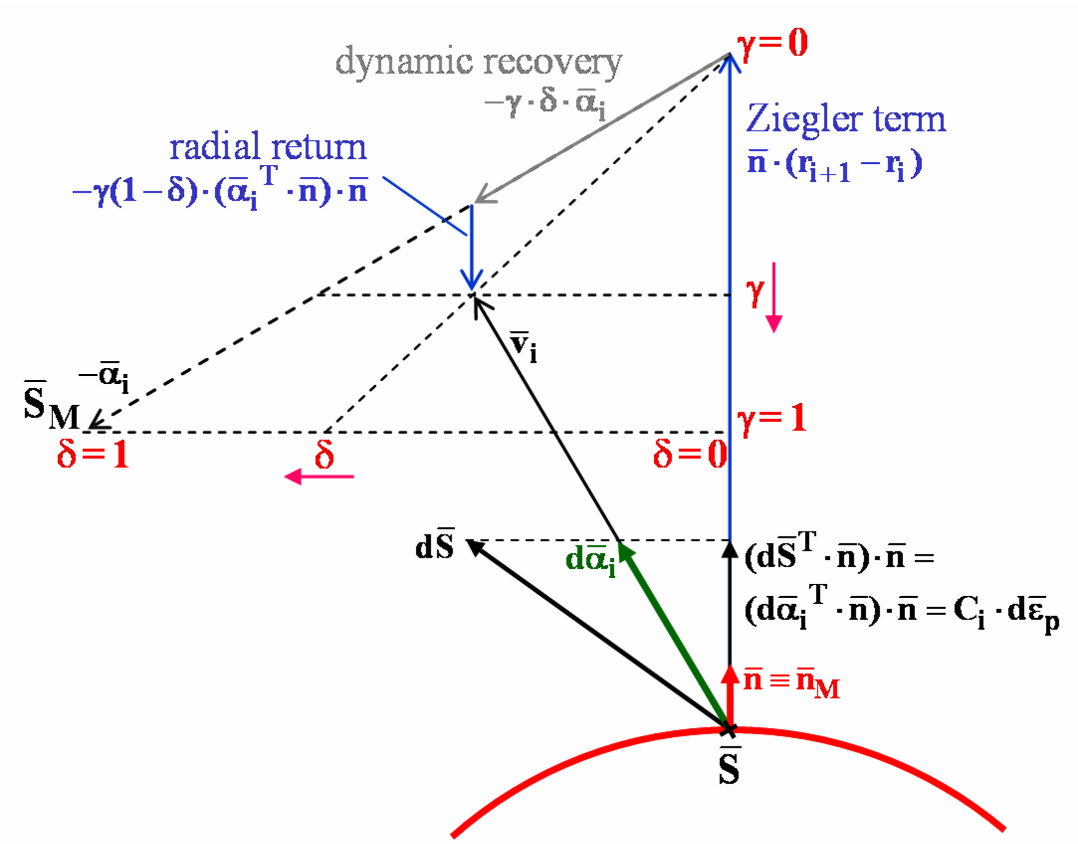

Fig. 3: Calculation of the translation direction $\bar{\mathbf{v}}_i$ of the hardening surfaces as a function of the material parameters γ and δ.used to model the radial return and dynamic recovery terms.

Finally, the translation rule for the surface centers $\bar{\mathbf{S}}_{\mathbf{c_i}}$ becomes

$$\mathbf{d\bar{S}_{c_i} - d\bar{S}_{c_{i+1}} = d\bar{\alpha}_i = C_i \cdot \bar{v}_i \cdot | d\bar{\varepsilon}_p |} \tag{7}$$

where $\mathbf{d\bar{\varepsilon}_p}$ is a plastic strain increment.

Such generalized approach is able to reproduce the Jiang-Sehitoglu [6], Ohno-Wang [7], Burlet-Cailletaud-Geyer (BCG) [8], Chaboche [3] and Armstrong-Frederick [4] NLK models, by choosing the appropriate values of γ and δ. Mròz [1] and Garud's [2] models are also reproduced, as long as the above equations are not applied to translate the surfaces outside the active one.

CONCLUSIONS

The proposed framework is able to reproduce the Mròz and Garud multi-surface models and all non-linear kinematic (NLK) models using the same notation and variables, therefore providing a unified approach to directly compare them and even to mix their surface translation rules. The combination of NLK rules with multi-surface capability efficiently deals with stress-strain predictions under non-proportional loadings even for very long and complex variable amplitude histories.

REFERENCES

[1] Mroz, Z.:
On the description of anisotropic work hardening. J Mech Phys Solids 15, p.163-175 (1967).

[2] Garud, Y.S.:
A New Approach to the Evaluation of Fatigue under Multiaxial Loading. J of Engineering Materials and Technology, ASME, vol. 103, pp.118-125 (1981).

[3] Chaboche, J.L., Dang-Van, K., Cordier, G.:
Modelization of the strain memory effect on the cyclic hardening of 316 stainless steel. In: SMIRT 5, Berlin (1979).

[4] Armstrong, P.J., Frederick, C.O.:
A mathematical representation of the multiaxial Bauschinger effect, CEGB Report RD/B/N731, Berkeley Nuclear Lab (1966).

[5] Ohno, N., Wang, J.D.:
Transformation of a nonlinear kinematic hardening rule to a multisurface form under isothermal and nonisothermal conditions. Int. J. Plasticity 7, p.879-891 (1991).

[6] Jiang, Y., Sehitoglu, H.:
Modelling of cyclic ratchetting plasticity, part I: development of constitutive relations. ASME J. Engng. Mat. and Techn. 63, p.720-725 (1996).

[7] Ohno, N., Wang, J.D.:
Kinematic hardening rules with critical state of dynamic recovery, Part I: formulation and basic features for ratchetting behavior. Int. J. Plasticity 9, p.375-390 (1993).

[8] Burlet, H., Cailletaud, G.:
Modelling of cyclic plasticity in finite element codes, Proc. of Constitutive Laws for Engineering Materials: Theory and Applications, p.1157-1164 (1987).

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