GENERALIZATION OF THE MOMENT OF INERTIA METHOD TO PREDICT EQUIVALENT AMPLITUDES OF NON-PROPORTIONAL MULTIAXIAL STRESS OR STRAIN HISTORIES

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Abstract. *Several multiaxial fatigue damage models have been introduced in the literature. Most of them require some measure of an equivalent stress or strain amplitude, which may be difficult to obtain for non-proportional (NP) multiaxial load histories. To identify individual cycles, a multiaxial rainflow algorithm must be employed. Then, for each rainflow-counted cycle, the equivalent stress or strain amplitude of its path is often computed using the so-called convex enclosure methods, which try to find spheres, ellipsoids or rectangular prisms that contain such path in a deviatoric space. However, such procedure involves information loss, in special if the path shape is very different from the shape of the enclosing convex surface, resulting in poor estimates of equivalent amplitudes. To overcome this problem, the Moment of Inertia (MOI) method has been proposed to calculate equivalent amplitudes and mean components of two-dimensional (2D) stress or strain paths, generated e.g. by tension-torsion or biaxial histories. In this work, the MOI method is extended to general 6D stress or strain paths, which include all normal and shear components. To accomplish that, the history is represented as a path in a 5D deviatoric stress or strain space. This 5D path is then assumed to be a homogeneous wire with unit mass, whose perimeter centroid is used to estimate the location of the path mean component. Then, the Polar Moment of Inertia (PMOI) of such hypothetical wire with respect to its mean component is calculated, which represents the distribution of the path about a single point, the perimeter centroid. Thus, the PMOI gives a measure of how much the path stretches away from its mean component, which is used in the calculation of the equivalent amplitudes. Experimental results for 13 different multiaxial histories prove the effectiveness of the proposed method to predict equivalent amplitudes and fatigue lives.*

Keywords*: multiaxial fatigue, non-proportional loadings, equivalent amplitude, moment of inertia method, deviatoric space*

1. INTRODUCTION

As explained in Socie and Marquis (1999), several multiaxial fatigue damage models have been introduced in the literature, such as the ones proposed by Sines, Crossland, Findley, McDiarmid, Brown-Miller, Fatemi-Socie and Smith-Watson-Topper (SWT). All of them require some measure of an equivalent stress or strain range, which may be difficult to obtain for non-proportional (NP) multiaxial load histories.

For a given multiaxial stress-strain NP history, the fatigue damage can be calculated by projecting the history onto a candidate plane at the critical point. This critical plane approach is simple to compute for Case A cracks, which initiate perpendicular to the free surface. In this case, the in-plane shear stress or strain may be counted using a uniaxial rainflow algorithm. On the other hand, for Case B cracks, which initiate at a 45^o angle from the free surface, a multiaxial rainflow count must be performed to identify individual cycles formed by the in-plane and out-of-plane shear components.

For each rainflow-counted cycle, the equivalent stress or strain range is often computed using the so-called convex enclosure methods, which try to find circles, ellipses or rectangles that contain the entire projected path in the 2D case, or hyperspheres, hyperellipsoids or hyperprisms in a generic 5-dimensional (5D) equivalent stress space. The traditional convex enclosure methods have been reviewed by Meggiolaro and Castro (2012): the Minimum Ball, Minimum Circumscribed Ellipsoid, Minimum Volume Ellipsoid, Minimum F-norm Ellipsoid (MFE), Maximum Prismatic Hull and Maximum Volume Prismatic Hull. These methods make use of stress and strain parameters such as the von Mises stress and strain ranges $\Delta\sigma_{Mises}$ and $\Delta\varepsilon_{Mises}$, defined by:

$$\Delta\sigma_{Mises} = \frac{\sqrt{(\Delta\sigma_x - \Delta\sigma_y)^2 + (\Delta\sigma_x - \Delta\sigma_z)^2 + (\Delta\sigma_y - \Delta\sigma_z)^2 + 6(\Delta\tau_{xy}^2 + \Delta\tau_{xz}^2 + \Delta\tau_{yz}^2)}}{\sqrt{2}} \quad (1)$$

$$\Delta\varepsilon_{Mises} = \frac{\sqrt{(\Delta\varepsilon_x - \Delta\varepsilon_y)^2 + (\Delta\varepsilon_x - \Delta\varepsilon_z)^2 + (\Delta\varepsilon_y - \Delta\varepsilon_z)^2 + 1.5(\Delta\gamma_{xy}^2 + \Delta\gamma_{xz}^2 + \Delta\gamma_{yz}^2)}}{\sqrt{2}\cdot(1+\bar{\nu})} \tag{2}$$

where the $\bar{\nu}$ is the mean (or effective) Poisson coefficient $\bar{\nu} = (0.5\varepsilon_p + \nu_e\varepsilon_e)/(\varepsilon_p + \varepsilon_e)$, while ε_e and ε_p are the elastic and plastic components of the strains, and ν_e and ν_p are the elastic and plastic Poisson coefficients ($\nu_p = 0.5$ assuming plastic strains conserve material volume).

However, extensive simulations showed that all convex enclosure methods can lead to poor predictions of the *mean* stresses or strains, if they are assumed as located at the center of the ball, ellipse or prism, which shows a stress path shaped very differently from an ellipse and its Minimum F-norm Ellipsoid (MFE) enclosure. Convex enclosure methods may also result in poor estimates of stress or strain *amplitudes*, in special for highly non-convex NP history paths, such as cross or star-shaped paths.

The Moment Of Inertia (MOI) method has been proposed by Meggiolaro and Castro (2012, 2013) initially to calculate alternate and mean components of two-dimensional (2D) non-proportional load histories. To accomplish that, the history must first be represented in a 2D Euclidean sub-space of the transformed 5D deviatoric stress-space $E_{5\sigma}$ (for stress histories) or strain-space $E_{5\varepsilon}$ (for strain histories). These 5D deviatoric spaces represent the stress and strain states using the vectors $\bar{S}'$ and $\bar{e}'$, defined as

$$\bar{S}' \equiv [S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5]^T \text{ and } \bar{e}' \equiv [e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5]^T \tag{3}$$

where

$$S_1 \equiv \sigma_x - \frac{\sigma_y + \sigma_z}{2}, \quad S_2 \equiv \frac{\sigma_y - \sigma_z}{2}\sqrt{3}, \quad S_3 \equiv \tau_{xy}\sqrt{3}, \quad S_4 \equiv \tau_{xz}\sqrt{3}, \quad S_5 \equiv \tau_{yz}\sqrt{3} \tag{4}$$

$$e_1 \equiv \varepsilon_x - \frac{\varepsilon_y + \varepsilon_z}{2}, \quad e_2 \equiv \frac{\varepsilon_y - \varepsilon_z}{2}\sqrt{3}, \quad e_3 \equiv \frac{\gamma_{xy}}{2}\sqrt{3}, \quad e_4 \equiv \frac{\gamma_{xz}}{2}\sqrt{3}, \quad e_5 \equiv \frac{\gamma_{yz}}{2}\sqrt{3} \tag{5}$$

Note that the 5D stress-space used in the MOI method is a scaled version of the Euclidean space proposed by Papadopoulos et al. (1997). The MOI method assumes that the 2D load path, represented by a series of points (X, Y) that describe the stress or strain variations along it, is analogous to a homogeneous wire with unit mass. Note that X and Y can have stress or strain units, but they are completely unrelated to the directions x and y usually associated with the material surface. The mean component of the path is assumed, in the MOI method, to be located at the center of gravity of this hypothetical homogeneous wire shaped as the load history path. Such center of gravity is located at the perimeter centroid (X_c, Y_c) of the path, calculated from contour integrals along it

$$X_c = \frac{1}{p_{XY}}\cdot\oint X\cdot dp_{XY}, \quad Y_c = \frac{1}{p_{XY}}\cdot\oint Y\cdot dp_{XY}, \quad p_{XY} = \oint dp_{XY} \tag{6}$$

where dp_{XY} is the length of an infinitesimal arc of the path and p_{XY} is the path perimeter, see Fig. 1.

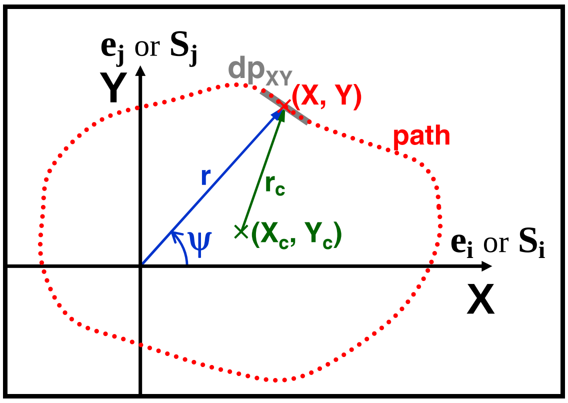

Figure 1. History path, assumed as a homogeneous wire with unit mass.

Note that this perimeter centroid (PC) is in general different from the area centroid (AC), which is the center of gravity of a uniform density sheet bounded by the shape of the closed stress or strain path. The reason to choose the perimeter centroid instead of the area centroid to locate the mean component can be readily seen in the example in Fig. 2. In this example, the right portion of the history has almost zero area, therefore it allows the area centroid to be located approximately at the origin of the diagram, which is not physically reasonable for such an asymmetrical path. The perimeter centroid, on the other hand, gives a much better estimate of the mean component of such stress or strain paths.

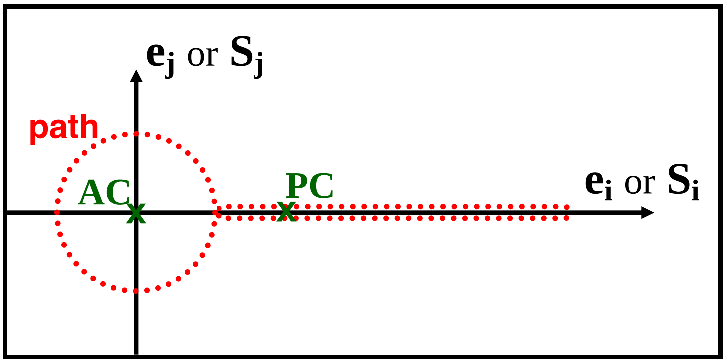

Figure 2. The area centroid (AC) does not reflect well the mean component of the stress or strain path, while the perimeter centroid (PC) is a good measure of such mean.

As described in (Meggiolaro and Castro, 2012), the MOI method calculates the equivalent stress or strain ranges of a loading path from the mass moments of inertia (MOI) of its analogous homogeneous wire, calculated with respect to its perimeter centroid. These MOI at the centroid are obtained from the parallel axis theorem applied to the MOI calculated at the origin O of the diagram, given by the contour integrals along the path

$$I_{XX}^{O} = \frac{1}{p_{XY}} \cdot \oint Y^2 \cdot dp_{XY}, \quad I_{YY}^{O} = \frac{1}{p_{XY}} \cdot \oint X^2 \cdot dp_{XY}, \quad I_{XY}^{O} = I_{YX}^{O} = -\frac{1}{p_{XY}} \cdot \oint X \cdot Y \cdot dp_{XY} \tag{7}$$

2. MOI METHOD FOR GENERAL LOAD HISTORIES

In this work, the Moment Of Inertia (MOI) method is generalized to calculate the alternate and mean components of a load path involving all six stress or strain components, instead of only two. When a 6D stress or strain path is represented in the 5D deviatoric sub-space associated with $\vec{S}'$ or $\vec{e}'$, the stress or strain scalar quantities associated with its Polar MOI can be defined as

$$I_p \equiv \frac{1}{p_S} \cdot \oint \underbrace{|\vec{S}' - \vec{S}_m'|}_{r_m}{}^2 \cdot dp_S \quad \text{or} \quad I_p \equiv \frac{1}{p_e} \cdot \oint \underbrace{|\vec{e}' - \vec{e}_m'|}_{r_m}{}^2 \cdot dp_e \tag{8}$$

where $dp_S \equiv |d\vec{S}'|$ and $dp_e \equiv |d\vec{e}'|$ are the lengths of infinitesimal arcs of the stress and strain paths, p_S and p_e are the respective path perimeters, the mean component $\vec{S}_m'$ or $\vec{e}_m'$ of the deviatoric path is given by

$$\vec{S}_m' = \frac{1}{p_S} \cdot \oint \vec{S}' \cdot dp_S, \quad p_S \equiv \oint dp_S \quad \text{or} \quad \vec{e}_m' = \frac{1}{p_e} \cdot \oint \vec{e}' \cdot dp_e, \quad p_e \equiv \oint dp_e \tag{9}$$

and r_m is the distance between each point in the path and its centroid.

To compute the actual mean stresses and strains, it is necessary to include the contribution of the mean hydrostatic stress σ_{hm} and strain ε_{hm}. From the matrix transformations between the 6D and 5D representations (Meggiolaro and Castro, 2012), the actual mean stress and strain components become

$$\vec{\sigma}_m = \frac{2}{3} A^T \cdot \vec{S}_m' + \vec{\sigma}_{hm} \quad \text{and} \quad \vec{\varepsilon}_m = \frac{2}{3} A^T \cdot \vec{e}_m' + \vec{\varepsilon}_{hm} \tag{10}$$

where

$$A = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3/2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3/2} \end{bmatrix} \tag{11}$$

$$\vec{\sigma}_{hm} = \sigma_{hm} \cdot [1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T \quad \text{and} \quad \vec{\varepsilon}_{hm} = \varepsilon_{hm} \cdot [1 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T \tag{12}$$

Analogously to the 2D version of the MOI method, if I_p is integrated along a *full* loading cycle, then the equivalent stress or strain amplitudes become

$$\frac{\Delta\sigma_{Mises}}{2} \text{ or } \frac{\Delta\varepsilon_{Mises}}{2} = \sqrt{3 \cdot I_p} \tag{13}$$

Note that, for 2D histories such as tension-torsion load paths, the equivalent stresses are identical to the ones from the original 2D MOI method (Meggiolaro and Castro, 2012). Therefore, this 5D approach is indeed a generalization of the 2D version. In addition, since the norm $|\bar{S}' - \bar{S}_m'|$ is equal to the relative Mises stress between each point $\bar{S}'$ from the path and the path centroid $\bar{S}_m'$, it is possible to interpret the MOI method for stress ranges as a root mean square (RMS) of relative Mises stresses along the path with respect to the centroid. Analogously, the MOI method for strain ranges can be interpreted as a RMS of the relative Mises strains $|\bar{e}' - \bar{e}_m'|$ along the path.

3. COMPARISONS AMONG THE EFFECTIVE RANGE PREDICTIONS

The MOI predictions of effective ranges along with the predictions from the Minimum Ball method (Dang Van and Papadopoulos, 1999) and Maximum Prismatic Hull method (Mamiya et al., 2009) are now compared to experimental measurements from Itoh (Kida et al., 1997) in a 304 stainless steel. Thirteen periodic histories are studied, represented by the block loadings shown in Fig. 3 for Cases 0 through 12. Note that most loadings from Fig. 3 consider 1 cycle per block, except for Cases 1 through 4, which consider 2 cycles per block. The number of cycles in each load block can be deterministically obtained using e.g. the Wang-Brown rainflow algorithm (Wang and Brown, 1996).

The predicted effective ranges in each case are used to calculate the multiaxial fatigue lives for each history, which are then compared to the experimental measurements. The Ramberg-Osgood stress-strain equations and the Smith-Watson-Topper (SWT) damage model are used in a critical plane approach to calculate the fatigue lives N (in cycles), using the properties of the 304 stainless steel:

$$\begin{aligned} &\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{1754}\right)^{1/0.276}, \; E = 197{,}000\ MPa, \; b = -0.0886, \; c = -0.303 \\ &(\sigma_{max} \frac{\Delta\varepsilon}{2})_{max} = \frac{757^2}{E}(2N)^{2b} + 30.5 \cdot (2N)^{b+c} \end{aligned} \tag{12}$$

Table 1 shows the life predictions according to the MOI, MB and MPH methods. Note that the MOI method considers 2 cycles per block for Cases 1 through 4 (marked with an * in the Table), as obtained by a Wang-Brown multiaxial rainflow count.

The MOI method predicts that Cases 0 through 5 are essentially proportional. This is reasonable, because the star and cross shaped histories from Cases 1-4 are indeed the combination of 2 perpendicular proportional paths. The MPH generates bad predictions in this case, since such method wouldn't be able to distinguish between a cross shaped and a circular history. For Cases 5-12, the MOI method also results in good predictions, which agree with the MPH method. However, the MB method implicitly assumes that all cases are proportional, leading to poor predictions for Cases 8-12.

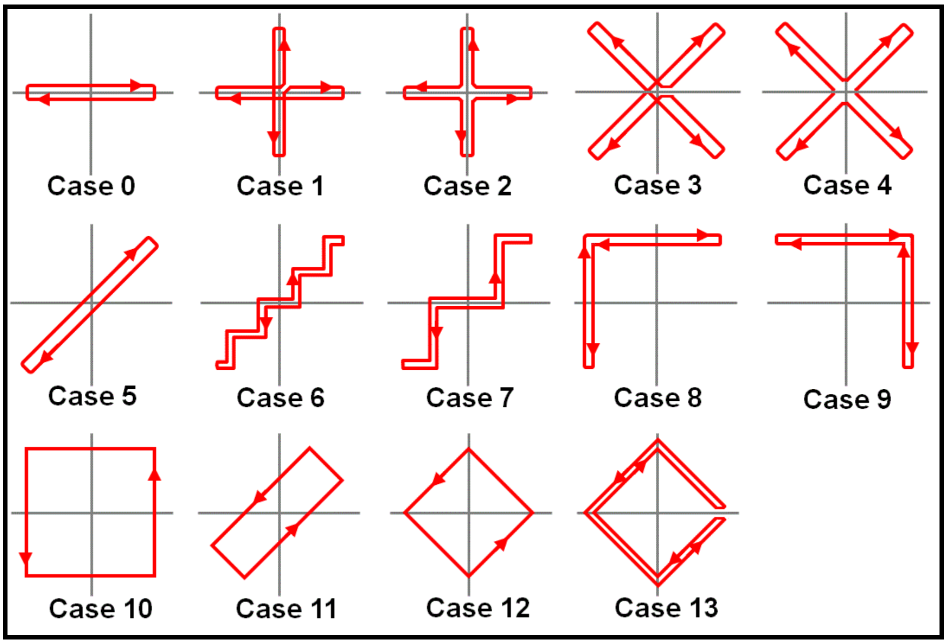

Figure 3. History paths used in the experimental validation of the F_{np} and equivalent range predictions.

Table 1. Fatigue life predictions *N* (in cycles) using the SWT damage model and the MOI, MB and MPH methods. Note that Cases 1-4 consider 2 cycles per block, e.g. the measured life for Case 1 was 1,400 blocks and thus 2,800 cycles.

path / N	experim.	MOI	MB	MPH
Case 0	7100	7085	7085	7085
Case 1	2800	3379	3379	1150
Case 2	4200	4462	4462	1504
Case 3	820	640	640	229
Case 4	900	858	858	304
Case 5	3200	3557	3557	3557
Case 6	2600	2332	2393	2177
Case 7	1700	1590	1751	1453
Case 8	470	604	856	572
Case 9	660	604	856	572
Case 10	320	329	949	329
Case 11	1200	1073	2241	1073
Case 12	710	689	2023	689

as if 90° out of phase (MPH, Cases 1-4)

as if proportional (MB, Cases 8-12)

In summary, the MOI method results in quite good life predictions in all studied histories, all estimated within only 20% from the experimental results, see Fig. 4. Note that these are not curve fittings, they are true predictions made using the MOI method (together with SWT) without any adjustable parameter. The MPH method, on the other hand, gives poor life predictions for Cases 1-4, since it wrongfully assumes that these cross or star-shaped histories are 90° out-of-phase, instead of being proportional. And the MB method results in non-conservative predictions in Cases 8-12, since it wrongfully assumes that these paths are proportional.

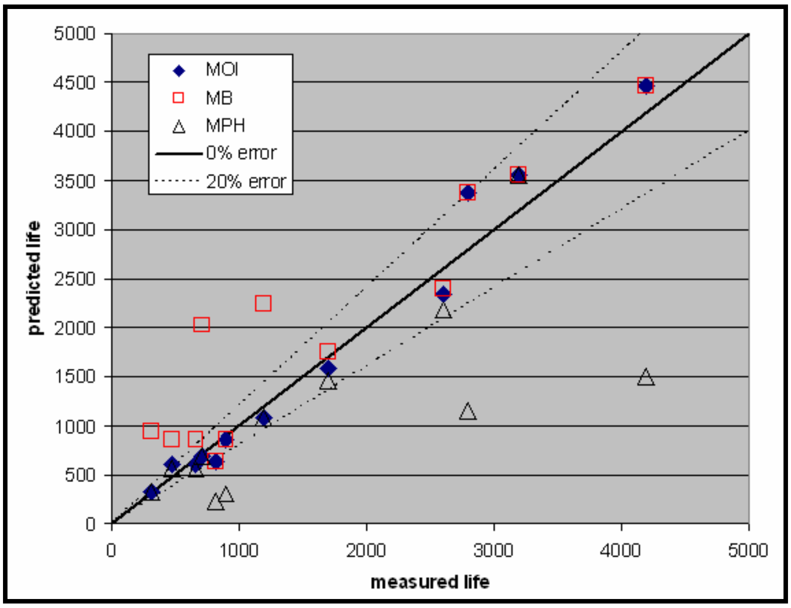

Figure 4. Measured and predicted fatigue lives using SWT and the MOI, MB and MPH methods, for a 304 steel.

4. CONCLUSIONS

The MOI method is able to obtain equivalent amplitude/range and mean components of non-proportional multiaxial histories, without the need for adjustable parameters or incremental plasticity calculations. The MOI method accounts for the contribution of every single segment of the stress or strain path, dealing with an arbitrarily shaped history without losing information about such shape. Experimental results demonstrated the effectiveness of the MOI method.

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