A Unified Rule to Estimate Multiaxial Elastoplastic Notch Stresses and Strains under In-Phase Proportional Loadings

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ABSTRACT. *Several methods have been proposed to estimate elastoplastic notch-tip stresses and strains from linear elastic calculations, the so-called elastoplastic notch correction. For uniaxial load histories, Neuber's and Glinka's rules are perhaps the most used. For non-proportional multiaxial histories, such corrections require incremental plasticity calculations to correlate stresses and strains at the notch root, a quite challenging task. But for in-phase proportional multiaxial histories, where the principal directions do not change and the load path in a stress diagram follows a straight line, approximate methods can be used without requiring an incremental approach. Most of these methods are based on Neuber's rule, which results in conservative predictions especially under plane strain-dominated cases associated with sharp notches. In this work, a Unified Notch Rule (UNR) is proposed for uniaxial and in-phase proportional multiaxial histories. The UNR can reproduce Neuber's or Glinka's rules, and even interpolate their notch-tip behaviors, or extrapolate them for notches with increased constraint. Moreover, the UNR also allows a non-zero normal stress perpendicular to the free-surface. The proposed method is compared with elastoplastic Finite Element calculations on notched shafts.*

INTRODUCTION

To calculate the elastoplastic (EP) strains from a given multiaxial stress history, it is usually necessary to adopt an incremental plasticity formulation, which integrates non-linear differential equations to calculate the stress-strain behavior [1]. In the presence of notches, a much simpler approach is to perform a single linear elastic (LE) Finite Element (FE) calculation on the entire piece for a static unit value of each applied loading. The resulting values at the notch root are called pseudo-stresses and pseudo-strains, which are fictitious quantities calculated using the theory of elasticity at the critical point of the piece, while assuming that the material follows Hooke's law [2]. These pseudo values are represented here with a "~" symbol on top of each variable.

Under in-phase proportional loadings, approximate models to obtain the stress and strain concentration factors K_σ and K_ε can be used to avoid computationally-intensive incremental plasticity calculations. They provide notch corrections that try to correlate pseudo and notch-tip values using a scalar parameter such as the von Mises equivalent stress. The main elastoplastic notch models for in-phase proportional histories are the constant ratio [3], Hoffmann-Seeger's [4-5], and Dowling's [6] models. These models require some variable definitions, namely:

- $\tilde{\sigma}_i$ and $\tilde{\varepsilon}_i$: pseudo principal stresses and strains at the notch tip, where $i = 1, 2, 3$.
- σ_i and ε_i: actual elastoplastic principal stresses and strains at the notch tip.
- λ_2 and λ_3: biaxiality ratios between principal stresses, $\lambda_2 \equiv \sigma_2/\sigma_1$ and $\lambda_3 \equiv \sigma_3/\sigma_1$, both assumed between -1 and 1.
- ϕ_2 and ϕ_3: biaxiality ratios between principal strains, where $\phi_2 \equiv \varepsilon_2/\varepsilon_1$ and $\phi_3 \equiv \varepsilon_3/\varepsilon_1$, also assumed between -1 and 1; and
- $\bar{\nu}$: effective Poisson ratio, with $\nu < \bar{\nu} \leq 0.5$ in the EP case.

Dowling's model [6] assumes that the principal stresses σ_1 and σ_2 act on the free surface of the critical point (thus $\sigma_3 = 0$), but it considers that both λ_2 and ϕ_2 are constant, estimating them from the pseudo-stresses and pseudo-strains:

$$\lambda_2 = \frac{\sigma_2}{\sigma_1} \cong \frac{\tilde{\sigma}_2}{\tilde{\sigma}_1} \cong \frac{\phi_2 + \nu}{1 + \phi_2 \nu}, \quad \phi_2 = \frac{\varepsilon_2}{\varepsilon_1} \cong \frac{\tilde{\varepsilon}_2}{\tilde{\varepsilon}_1} \cong \frac{\lambda_2 - \nu}{1 - \lambda_2 \nu} \tag{1}$$

The model then directly correlates σ_1 and ε_1 using effective Ramberg-Osgood parameters E^* and H_c^*:

$$\varepsilon_1 = \frac{\sigma_1}{E^*} + \left[\frac{\sigma_1}{H_c^*}\right]^{1/h_c} \tag{2}$$

$$E^* = E \cdot \left(\frac{1 + \phi_2 \nu}{1 - \nu^2}\right), \quad H_c^* = H_c \cdot \frac{(1 - \lambda_2 + \lambda_2^2)^{(h_c - 1)/2}}{(1 - \lambda_2 / 2)^{h_c}} \tag{3}$$

In notched components, assuming that the principal directions of the EP stresses and pseudo-stresses are equal, a variation of Neuber's rule [7] could be used to calculate the EP notch-tip σ_1 (and then ε_1) from the pseudo $\tilde{\sigma}_1$:

$$\tilde{\sigma}_1 \cdot \left[\frac{\tilde{\sigma}_1}{E^*}\right] = \tilde{\sigma}_1 \cdot \tilde{\varepsilon}_1 = \sigma_1 \cdot \varepsilon_1 = \sigma_1 \cdot \left[\frac{\sigma_1}{E^*} + \left(\frac{\sigma_1}{H_c^*}\right)^{1/h_c}\right] \; (Dowling) \tag{4}$$

The above equation does not require a plastic term on the left hand side, because the pseudo-stresses and pseudo-strains are, by definition, LE. Finally, the other notch-tip EP principal stresses and strains are then obtained from σ_1 and ε_1:

$$\begin{cases} \sigma_2 = \lambda_2 \sigma_1, \quad \sigma_3 = 0 \\ \varepsilon_2 = \phi_2 \varepsilon_1, \quad \varepsilon_3 = -\bar{\nu} \varepsilon_1 \dfrac{1 + \lambda_2}{1 - \lambda_2 \bar{\nu}}, \quad \bar{\nu} = 0.5 - (0.5 - \nu) \dfrac{\sigma_1}{E^* \varepsilon_1} \end{cases} \tag{5}$$

THE UNIAXIAL UNIFIED NOTCH RULE (UNR)

Noting that Glinka's rule [8] usually underestimates while Neuber's rule [7] overestimates notch-tip stresses and strains, when compared to experimental results and FE analyses, a unified incremental rule has been proposed in [9], which returns values in-between them. For a monotonic uniaxial loading in the *x* direction, it states that

$$\sigma_x d\varepsilon_x \cdot (1+\alpha_{ED}) + \varepsilon_x d\sigma_x \cdot (1-\alpha_{ED}) = \tilde{\sigma}_x d\tilde{\varepsilon}_x + \tilde{\varepsilon}_x d\tilde{\sigma}_x \tag{6}$$

where $0 \le \alpha_{ED} \le 1$ was called the *energy dissipation coefficient*, assumed in [9] as a material parameter, estimated from $\alpha_{ED} \cong (1 - 2h_c)/(1 - h_c)$ based on an energy argument, where h_c is the cyclic exponent of Ramberg-Osgood's equation. However, α_{ED} might depend not only on the material but also on the notch geometry and constraint factor. This coefficient α_{ED} can also be regarded as a fitting parameter if experimental data or reliable EP FE analyses are available for its calibration.

A deviatoric version of Eq. 6 is proposed in this work:

$$s_x de_x \cdot (\alpha_U) + e_x ds_x \cdot (2-\alpha_U) = \tilde{s}_x d\tilde{e}_x + \tilde{e}_x d\tilde{s}_x \tag{7}$$

where $s_x \equiv (2\sigma_x - \sigma_y - \sigma_z)/3$ and $e_x \equiv (2\varepsilon_x - \varepsilon_y - \varepsilon_z)/3$ are deviatoric stresses and strains in the *x* direction, while $\alpha_U \equiv (1 + \alpha_{ED})$ is called the *notch constraint factor*, with values $1 \le \alpha_U \le 2$ to interpolate the Incremental Neuber rule [10-11] ($\alpha_U = 1$) and an Incremental Glinka rule ($\alpha_U = 2$).

As the deviatoric stresses s_x, s_y and s_z are linearly-dependent, since $s_x + s_y + s_z = 0$, it is possible to reduce the deviatoric stress and strain space dimensions using:

$$s_1 \equiv \sigma_x - \frac{\sigma_y + \sigma_z}{2} = \frac{3}{2} s_x \,, \quad s_2 \equiv \frac{\sigma_y - \sigma_z}{2}\sqrt{3} = \frac{s_y - s_z}{2}\sqrt{3} \tag{8}$$

$$e_1 \equiv \varepsilon_x - \frac{\varepsilon_y + \varepsilon_z}{2} = \frac{3}{2} e_x \,, \quad e_2 \equiv \frac{\varepsilon_y - \varepsilon_z}{2}\sqrt{3} = \frac{e_y - e_z}{2}\sqrt{3} \tag{9}$$

Assuming that Eq. 7 is valid for the transformed deviatoric stresses and strains from Eqs. 8 and 9, then

$$\begin{cases} (\alpha_U) \cdot s_1 de_1 + (2-\alpha_U) \cdot e_1 ds_1 = \tilde{s}_1 d\tilde{e}_1 + \tilde{e}_1 d\tilde{s}_1 \\ (\alpha_U) \cdot s_2 de_2 + (2-\alpha_U) \cdot e_2 ds_2 = \tilde{s}_2 d\tilde{e}_2 + \tilde{e}_2 d\tilde{s}_2 \end{cases} \tag{10}$$

where, as explained before, the symbol "~" is used for pseudo-values calculated from LE analyses.

The Unified Notch Rule (UNR) proposed in this work can then be obtained from the integration of Eq. 10, which can be used for both uniaxial and in-phase proportional histories. For uniaxial histories, this integration results in the scalar UNR:

$$\tilde{\varepsilon}^2 = \frac{\sigma}{E} \cdot \left[\frac{\sigma}{E} + \bar{\alpha}_U \cdot \left(\frac{\sigma}{H_c} \right)^{1/h_c} \right], \quad \bar{\alpha}_U \equiv \frac{\alpha_U + h_c(2-\alpha_U)}{1+h_c} \qquad (UNR) \tag{11}$$

where $\bar{\alpha}_U$ is the *effective notch constraint factor*. This equation can reproduce Neuber for $\alpha_U = 1$ and thus $\bar{\alpha}_U = 1$, or Glinka's rule for $\alpha_U = 2$ and thus $\bar{\alpha}_U = 2/(1+h_c)$.

Although different, α_U shares similarities with Newman's constraint factor α [12], varying from *1.0* under plane stress conditions (where Neuber's rule is recommended) to usually more than *3.0* under plane strain. Thus, both α_U and Newman's α reflect increased stress-state constraint and associated plasticity decrease at the critical point, however using α_U at notch tips and Newman's α at crack tips.

THE MULTIAXIAL UNIFIED NOTCH RULE

The multiaxial version of the UNR assumes in-phase proportional loading under free-surface conditions $\tau_{xz} = \tau_{yz} = 0$, but allows the presence of a surface normal $\sigma_z \neq 0$, where the z axis is assumed perpendicular to the surface, and the x and y axes aligned with the remaining principal directions, with x in the direction of the maximum absolute principal stress. Therefore, the principal stresses $\sigma_x \equiv \sigma_1$, $\sigma_y \equiv \sigma_2$, and $\sigma_z \equiv \sigma_3$ are assumed to satisfy $|\sigma_x| \geq |\sigma_y|$ and $|\sigma_x| \geq |\sigma_z|$ during the entire history. The involved variables are the same as the ones defined before, in addition to an elastic and plastic separation of the strain biaxiality ratios, through:

- ϕ_{2el} and ϕ_{3el}: biaxiality ratios between principal *elastic* strains, where $\phi_{2el} \equiv \varepsilon_{2el}/\varepsilon_{1el}$ and $\phi_{3el} \equiv \varepsilon_{3el}/\varepsilon_{1el}$, both assumed between -1 and 1; and
- ϕ_{2pl} and ϕ_{3pl}: same definition, but for *plastic* strains (for pressure-insensitive materials, where $\varepsilon_{1pl} + \varepsilon_{2pl} + \varepsilon_{3pl} = 0$, if follows that $1 + \phi_{2pl} + \phi_{3pl} = 0$ and thus $\phi_{2pl} + \phi_{3pl} = -1$).

Since the multiaxial loading history is assumed here to be proportional, the deviatoric stress increment is always parallel to the plastic straining direction, so the Prandtl-Reuss plastic flow rule [1] gives, for the normal deviatoric strain components,

$$\begin{bmatrix} de_1 \\ de_2 \end{bmatrix} = \begin{bmatrix} d\varepsilon_{xpl} - (d\varepsilon_{ypl} + d\varepsilon_{zpl})/2 \\ (d\varepsilon_{ypl} - d\varepsilon_{zpl}) \cdot \sqrt{3}/2 \end{bmatrix} = \frac{1}{P} \cdot \begin{bmatrix} d\sigma_x - (d\sigma_y + d\sigma_z)/2 \\ (d\sigma_y - d\sigma_z) \cdot \sqrt{3}/2 \end{bmatrix} \tag{12}$$

where P is called the *generalized plastic modulus* (proportional to the slope of the stress vs. plastic strain curve at the current stress state), and all shear increments are zero since x, y, and z are defined in the principal directions. Integrating the above equation using the plastic biaxiality ratio definitions, then

$$\int_0^{\varepsilon_{xpl}} d\varepsilon_{xpl} \cdot \begin{bmatrix} 1 - (\phi_{2pl} + \phi_{3pl})/2 \\ (\phi_{2pl} - \phi_{3pl}) \cdot \sqrt{3}/2 \end{bmatrix} = \int_0^{\sigma_x} \frac{1}{P} \cdot d\sigma_x \cdot \begin{bmatrix} 1 - (\lambda_2 + \lambda_3)/2 \\ (\lambda_2 - \lambda_3) \cdot \sqrt{3}/2 \end{bmatrix} \tag{13}$$

Neglecting the isotropic hardening transient, let's assume that the material follows Ramberg-Osgood with cyclic constant H_c and exponent h_c. Moreover, assuming that this proportional loading is balanced, i.e. it does not cause ratcheting or mean stress relaxation, then a Mróz multi-surface hardening model can be adopted [1] (instead of the more general non-linear kinematic hardening models). To improve accuracy, let's adopt an infinite number of hardening surfaces, as discussed in [13], see Fig. 1. From the calibration of the Mróz model, the generalized plastic modulus $P = P_i$ for the hardening surface with radius r_i becomes

$$P_i = (2/3) \cdot h_c H_c \left(r_i / H_c\right)^{1-1/h_c} \quad (14)$$

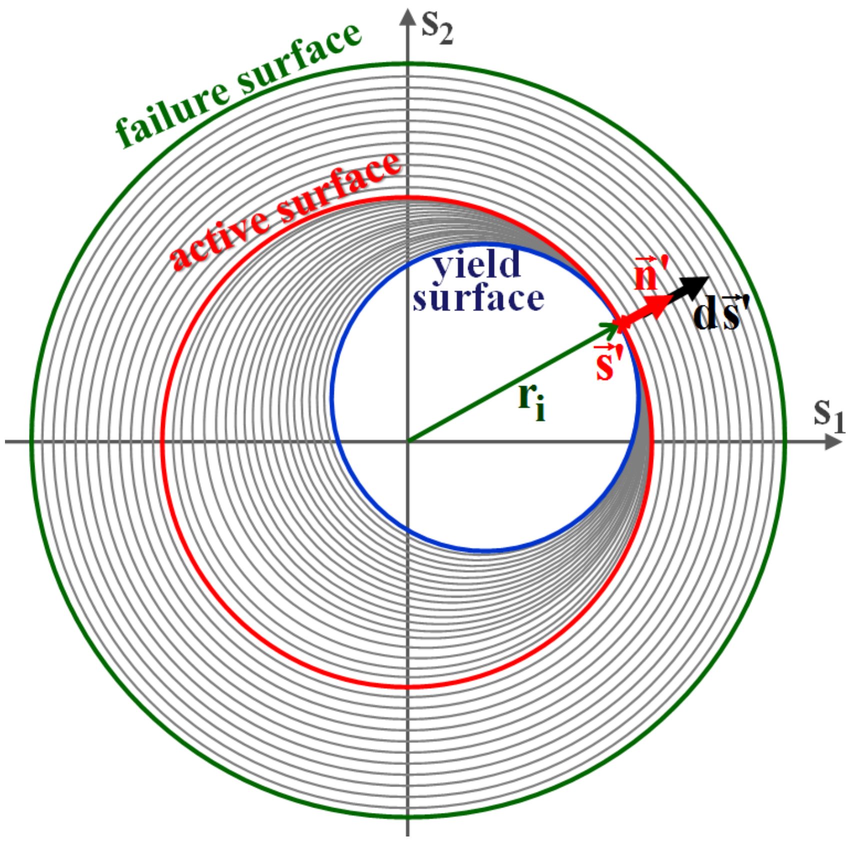

Fig. 1: Mróz infinite-surface hardening model for a monotonic proportional loading.

Consider a monotonic proportional loading departing from the origin of the deviatoric stress space, as shown in Fig. 1, assuming x, y and z as principal directions. In this case, the radius r_i of the current active surface from the Mróz model is equal to the norm (and thus the von Mises equivalent value) of the current stress state. Replacing the values of $P = P_i$ and r_i into Eq. 13, and using the plastic strain incompressibility condition $\phi_{2pl} + \phi_{3pl} = -1$, it follows that

$$\tilde{\sigma}_1 \cdot \left[\frac{\tilde{\sigma}_1}{E^*} \right] = \sigma_1 \cdot \left[\frac{\sigma_1}{E^*} + \bar{\alpha}_U \cdot \left(\frac{\sigma_1}{H_c^*} \right)^{1/h_c} \right] \tag{15}$$

$$E^* \equiv E/[1-\nu\cdot(\lambda_2+\lambda_3)]\ ,\ H_c^* \equiv H_c \cdot \frac{[1-(\lambda_2+\lambda_3)+(\lambda_2^2+\lambda_3^2)-\lambda_2\lambda_3]^{(h_c-1)/2}}{[1-(\lambda_2+\lambda_3)/2]^{h_c}} \tag{16}$$

$$\begin{cases} \varepsilon_{1el} = \sigma_1/E^*\ ,\ \ \varepsilon_{1pl} = \left(\sigma_1/H_c^*\right)^{1/h_c}\ ,\ \ \varepsilon_1 = \varepsilon_{1el} + \varepsilon_{1pl} \\ \sigma_2 = \lambda_2\sigma_1,\ \ \sigma_3 = \lambda_3\sigma_1 \\ \varepsilon_2 = \phi_{2el}\cdot\varepsilon_{1el} + \phi_{2pl}\cdot\varepsilon_{1pl},\ \ \ \varepsilon_3 = \phi_{3el}\cdot\varepsilon_{1el} + \phi_{3pl}\cdot\varepsilon_{1pl} \end{cases} \tag{17}$$

$$\phi_{2pl} = \frac{\lambda_2 - 0.5\cdot(1+\lambda_3)}{1-0.5\cdot(\lambda_2+\lambda_3)},\quad \phi_{3pl} = \frac{\lambda_3 - 0.5\cdot(1+\lambda_2)}{1-0.5\cdot(\lambda_2+\lambda_3)} \tag{18}$$

$$\phi_{2el} = \frac{\lambda_2 - \nu\cdot(1+\lambda_3)}{1-\nu\cdot(\lambda_2+\lambda_3)},\quad \phi_{3el} = \frac{\lambda_3 - \nu\cdot(1+\lambda_2)}{1-\nu\cdot(\lambda_2+\lambda_3)} \tag{19}$$

Dowling's model for in-phase proportional loadings is a particular case of the more general in-phase proportional UNR, setting $\bar{\alpha}_U = 1$ (to reproduce Neuber's rule) and also $\lambda_3 = 0$ (free-surface with $\sigma_3 = 0$), assuming as well that $\phi_{2pl} = \phi_{2el}$ based on ν, and that $\phi_{3pl} = \phi_{3el}$ based on an effective Poisson ratio $\bar{\nu}$.

Both Dowling's and UNR models assume the nominal section (away from the notch) remains LE. In other words, they are valid even under general yielding of the net cross section, but they do not account for yielding of the gross cross section. To perform this correction, the pseudo principal stress $\tilde{\sigma}_1$ is represented as the product of a LE stress concentration factor K_t multiplied by a nominal stress σ_{n1}, i.e. $\sigma_{n1} \equiv \tilde{\sigma}_1 / K_t$, where σ_{n1} is assumed to follow Ramberg-Osgood, giving

$$K_t^2 \cdot \sigma_{n1} \cdot \left[\frac{\sigma_{n1}}{E^*} + \bar{\alpha}_U \cdot \left(\frac{\sigma_{n1}}{H_c^*} \right)^{1/h_c} \right] = \sigma_1 \cdot \left[\frac{\sigma_1}{E^*} + \bar{\alpha}_U \cdot \left(\frac{\sigma_1}{H_c^*} \right)^{1/h_c} \right] \tag{20}$$

VERIFICATION OF THE UNR WITH ELASTOPLASTIC FINITE ELEMENTS

The proposed UNR and Dowling's classic notch rule are checked against elastoplastic (EP) Finite Element (FE) calculations, for multiaxial in-phase proportional tension-torsion problems. The comparison is based on the calculation of the peak EP stresses and strains at a notched solid shaft with largest diameter 2" and semi-circular U-notch with radius 0.5". The shaft is assumed made of a heat-treated 1070 steel with Young modulus $E = 210GPa$, Poisson ratio $\nu = 0.3$, and Ramberg-Osgood parameters $H_c = 1736MPa$ and $h_c = 0.199$, measured by [14]. Figure 2 shows the adopted FE mesh from the ANSYS software, using the SOLID186 3D elements with 20 nodes each and 3 degrees of freedom per node for the EP calculations.

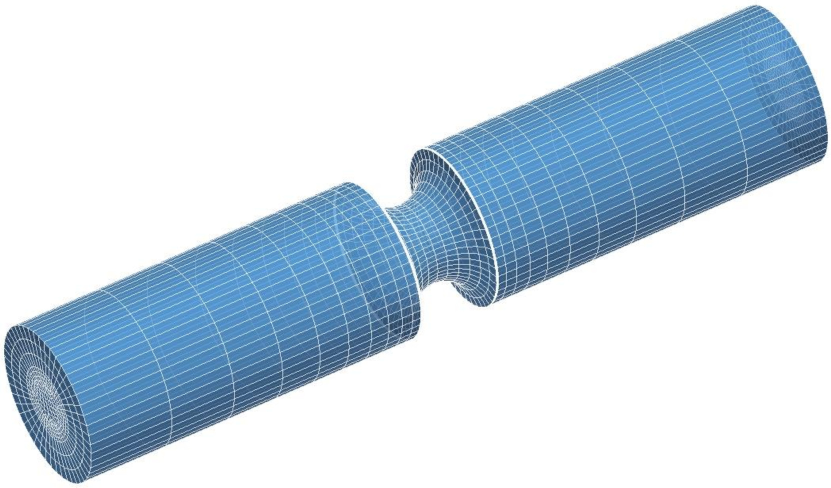

Figure 2: Adopted geometry and FE mesh for the EP calculations.

Figure 3 shows the strain K_ε and stress K_σ concentration factors for a particular case under pure torsion, where the LE $K_t = 1.17$, for several nominal shear stresses τ_n. It is important to note that the adopted τ_n are EP values, obtained for the predictions from Ramberg-Osgood and from the FE calculations from the EP shear stress acting on a notch-free shaft with same *net* cross section and applied torsional moment. In this way, it is possible to account for plasticity effects in the nominal region away from the notch. The square and triangular markers show the carefully obtained EP output, overestimated by Neuber's rule ($\alpha_U = 1$, the rule adopted in Dowling's multiaxial model) and underestimated by Glinka's rule ($\alpha_U = 2$). The dashed lines in-between Neuber's and Glinka's rule predictions are the better estimates obtained from the proposed UNR, calibrated for $\alpha_U = 1.4$. This shows that the proposed rule is able to effectively interpolate Neuber's and Glinka's models, improving its versatility for specimens with increased constraints. Finally, as expected, all predictions tend to the LE value $K_t = 1.17$ under low stresses.

CONCLUSIONS

In this work, a Unified Notch Rule (UNR) was proposed to predict elastoplastic stresses and strains at a notch root from linear elastic calculations, for uniaxial and in-phase proportional multiaxial histories. The UNR can interpolate Neuber's or Glinka's rules from the α_U parameter calibration, or even extrapolate them to better reproduce increased constraint effects. The UNR allows biaxiality ratios $\lambda_3 \equiv \sigma_3/\sigma_1 \neq 0$, an improvement over Dowling's model, which always assume $\lambda_3 = 0$. Even though the derivation of the UNR model assumed integration for a monotonic load, the resulting equations could be applied to cyclic loadings, as long as they are also in-phase and proportional, and the appropriate biaxiality ratios can be assumed constant.

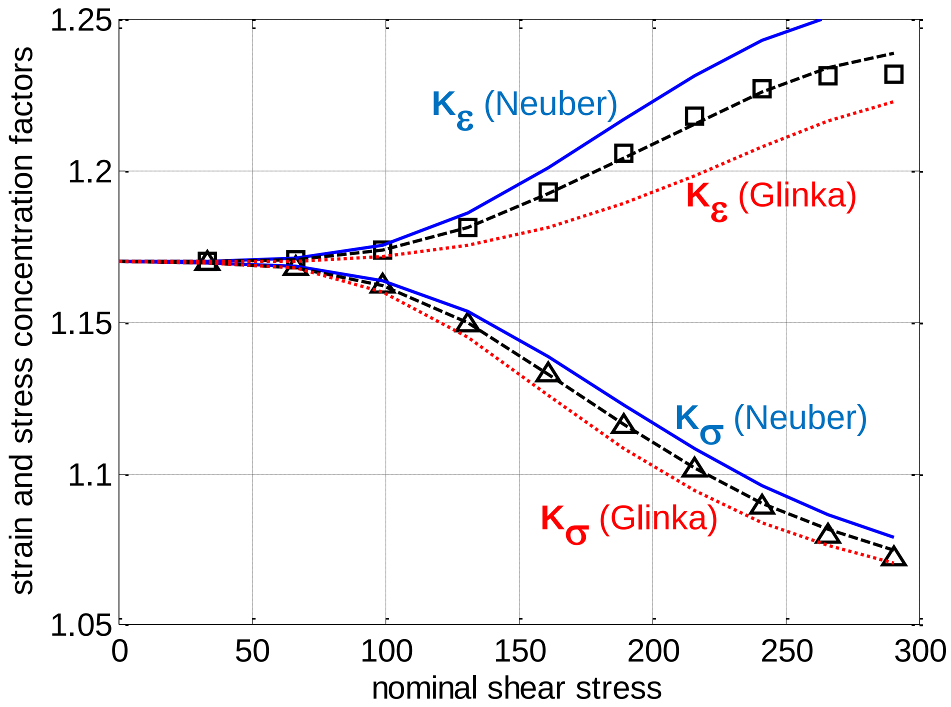

Figure 3: Predicted and FE-calculated EP strain and stress concentration factors.

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