A Multiaxial Incremental Fatigue Formulation using Nested Damage Surfaces

M.A. Meggiolaro[1], J.T.P. de Castro[1] and H. Wu[2]

[1] Pontifical Catholic University of Rio de Janeiro, PUC-Rio, Rua Marquês São Vicente, Rio de Janeiro, RJ, 22451-900, Brazil, meggi@puc-rio.br, jtcastro@puc-rio.br
[2] School of Aerospace Engineering and Applied Mechanics, Tongji University Siping Road 1239, 200092, Shanghai, P.R. China, wuhao@tongji.edu.cn

ABSTRACT. *Multiaxial fatigue damage calculation under non-proportional variable amplitude loading is a very challenging task, since traditional techniques require cycle identification and counting to single out individual load events. Moreover, to account for the non-proportionality of the load path of each event, semi-empirical methods are required to calculate path-equivalent ranges, e.g. using a convex enclosure or the MOI (Moment Of Inertia) method. In this work, a novel Incremental Fatigue methodology is introduced to continuously calculate the accumulation of multiaxial fatigue damage, without requiring rainflow counters or path-equivalent ranges. The proposed approach is not based on Continuum Damage Mechanics concepts or on the integration of elastoplastic work. Instead, fatigue damage itself is continuously integrated, based on traditional fatigue models using in engineering practice. A framework of nested damage surfaces is introduced, allowing the calculation of fatigue damage even for general 6D multiaxial load histories. The proposed approach is validated on non-proportional tension-torsion experiments on tubular 316L stainless steel specimens.*

INTRODUCTION

Most fatigue crack initiation models need to identify load cycles to compute the damage induced by them. This is because traditional fatigue models are discrete in nature, since they can accumulate damage only after a load event (e.g. a half-cycle) is properly identified, detected e.g. from a load reversal or from a hysteresis loop that closes. But the detection and counting of loading events can be a challenging task under multiaxial non-proportional (NP) histories. The existing multiaxial rainflow algorithms [1] are not robust, since they can output very different half-cycles depending on the choice of the initial counting point of a periodic load history [2]. Furthermore, fatigue damage computation requires the semi-empirical calculation of path-equivalent stress or strain ranges from the rainflow-counted paths [3].

On the other hand, a completely different fatigue calculation approach assumes damage as a continuous variable, with increments computed as the loading proceeds. Most works based on a continuous idea use Continuum Damage Mechanics concepts

[4], which need to be supplemented by purely phenomenological damage evolution equations that are difficult to calibrate, to say the least.

Other continuous damage approaches are based on an integration of elastoplastic work. However, the accumulated total work required to initiate a microcrack certainly is not a material property and still depends on the number of cycles, thus it is impossible to calculate without cycle and reversal detection, so it needs a rainflow counter.

Alternatively, instead of integrating dubious strain energy or energy-based damage parameters, the so-called Incremental Fatigue approach integrates fatigue damage itself, until reaching *1.0* or any other critical value using traditional accumulation concepts, as performed for the uniaxial case in [5-6]. Fatigue damage is thus continuously calculated after each infinitesimal stress or strain increment, not requiring the identification of load cycles. In this work, the Incremental Fatigue approach is extended to multiaxial fatigue, based on a direct analogy with non-linear incremental plasticity, however calculating damage instead of plastic strains.

INCREMENTAL FATIGUE APPROACH

The Incremental Fatigue (IF) approach was proposed for uniaxial histories in Wetzel and Topper's 1971 rheological model [5]. It makes use of the derivative of the normal stress σ with respect to damage D, called here *generalized damage modulus* D_σ, thus

$$D_\sigma \equiv d\sigma/dD \quad \Rightarrow \quad D = \int dD = \int (1/D_\sigma)\cdot d\sigma \qquad (1)$$

Consider a uniaxial constant amplitude loading history with stress amplitude σ_a. During a loading half-cycle, the excursion of the stress σ from $-\sigma_a$ to $+\sigma_a$ could be integrated according to Eq. 1 to find the associated fatigue damage $D = 1/2N$, however without explicitly calculating the fatigue life N. The damage D from this half-cycle is initially zero in the valley when $\sigma = -\sigma_a$ and thus $\Delta\sigma = \sigma-(-\sigma_a) = 0$, and continuously grows toward $D = 1/2N$ until σ reaches the peak $+\sigma_a$, when $\Delta\sigma = \sigma-(-\sigma_a) = 2\sigma_a$.

For simplicity, Wöhler's stress-based damage model is adopted below (strain-based models will be considered later). A simplified relation between the current stress state σ and the continuous damage D from the half-cycle excursion $-\sigma_a \rightarrow +\sigma_a$ can then be obtained from Wöhler's curve e.g. written in Basquin's notation:

$$\sigma_a = \sigma_c \cdot (2N)^b \quad \Rightarrow \quad \Delta\sigma/2 = [\sigma-(-\sigma_a)]/2 = \sigma_c/D^b \quad \Rightarrow \quad D = [(\sigma+\sigma_a)/2\sigma_c]^{-1/b} \qquad (2)$$

The generalized damage modulus D_σ during this half-cycle is such that

$$1/D_\sigma = dD/d\sigma = -[(\sigma+\sigma_a)/2\sigma_c]^{-1/b}/[b(\sigma+\sigma_a)] \qquad (3)$$

from which $D = 1/2N$ can be calculated using the integral

$$D=\int_{-\sigma_a}^{+\sigma_a} -\frac{1}{b(\sigma+\sigma_a)}\left(\frac{\sigma+\sigma_a}{2\sigma_c}\right)^{-1/b}\cdot d\sigma=\left(\frac{\sigma+\sigma_a}{2\sigma_c}\right)^{-1/b}\Bigg|_{-\sigma_a}^{+\sigma_a}=\left(\frac{\sigma_a}{\sigma_c}\right)^{-1/b}=\frac{1}{2N} \quad (4)$$

If this conceptually simple procedure could be generalized to multiaxial NP variable amplitude loading (VAL) histories, integrating damage along a general multiaxial load path, then cycle identification, multiaxial rainflow counting, and stress (or strain) range calculations would not be required to obtain the fatigue damage *D*. But this statement is easier said than done, since D_σ depends not only on the current stress state (σ in this uniaxial case), but also on the previous loading history (the value $-\sigma_a$ from the last reversal), see Eq. 3. So, Incremental Fatigue models need to allow D_σ to vary as a function of the stress level and of the existing state of damage [7].

The history dependence of D_σ, often neglected or overly simplified in the few IF models proposed in the literature, is analogous to the load-order dependence of hysteresis loops. Chu [6] outlined the generalization of Wetzel's rheological model to multiaxial loadings, indirectly detecting cycles using two simple rules. However, damage memory is not properly stored for NP VAL histories, where often no hysteresis loop actually closes and thus any virtual loop closure detection makes no sense.

MULTIAXIAL INCREMENTAL FATIGUE APPROACH

Stress-Based Incremental Fatigue Formulation

In this work, instead of using rheological models, a direct analogy between IF and incremental plasticity is adopted to store damage memory, using internal material variables. In incremental plasticity, a 5D deviatoric stress increment $d\vec{s}'$ can be used to calculate the associated 5D plastic strain increment $d\vec{e}'_{pl}$ from the current generalized plastic modulus *P*, using a plastic flow rule [8-9].

In particular, in the non-linear kinematic (NLK) incremental plasticity formulation, *plastic memory* is stored by the current arrangement among the hardening surfaces defined by their backstresses $\vec{\beta}'_i$, from which the *surface translation directions* $\vec{v}'_i$ are calculated (according to some translation rule) and combined with material coefficients p_i to calculate the current *P* [8-9]. No *plastic straining* occurs if the stress increment $d\vec{s}'$ happens inside the *yield surface,* whose radius should be equal or smaller than the *cyclic yield strength* S_{Yc}. The *accumulated plastic strain p* is then proportional to the integral of the scalar norm $|d\vec{e}'_{pl}|$ of the deviatoric plastic strain increments.

Let's now rephrase the previous paragraph for the desired IF model, based on the proposed direct analogy. In the IF model presented here, a 5D deviatoric stress increment $d\vec{s}'$ can be used to calculate the associated 5D *damage* increment $d\vec{D}'$ from the current *generalized damage modulus* D_σ, using a *damage evolution rule*. In the IF formulation, *damage memory* is stored by the current arrangement among *damage surfaces* defined by their *damage backstresses* $\vec{\beta}'_{\sigma i}$, from which the *damage surface* *translation directions* $\vec{v}'_{\sigma i}$ are calculated (according to some translation rule) and combined with material coefficients $d_{\sigma i}$ to calculate the current D_σ. No *damage* occurs if the stress increment $d\vec{s}'$ happens inside the *fatigue limit surface*, whose radius should be equal or smaller than the *fatigue limit* S_L. The *accumulated damage D* is then equal to the integral of the scalar norm $|d\vec{D}'|$ of the 5D damage increments.

The damage backstress vector $\vec{\beta}'_\sigma$ locates the center of the current fatigue limit surface, which can be decomposed as the sum of *M* damage backstresses $\vec{\beta}'_{\sigma 1}$, $\vec{\beta}'_{\sigma 2}$, …, $\vec{\beta}'_{\sigma M}$ that describe the relative positions between centers of consecutive damage surfaces, as illustrated in Fig. 1 for a 2D case.

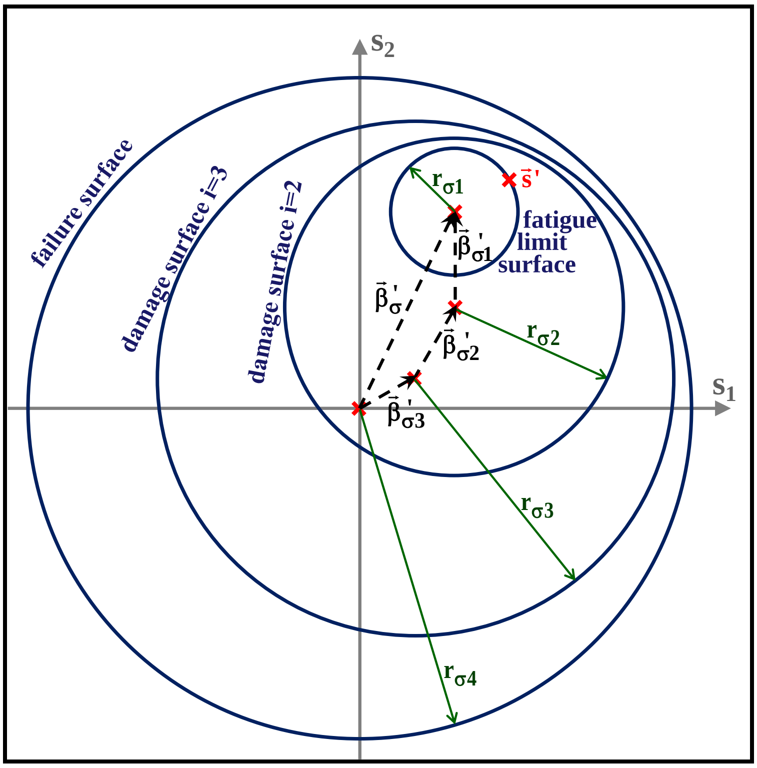

Figure 1. Fatigue limit, damage, and failure surfaces in a 2D deviatoric stress space for three moving surfaces, showing the damage backstress vector that defines the location of the fatigue limit surface center, and its three components that describe the relative positions between the centers of consecutive surfaces.

Each damage surface has a *constant* radius $r_{\sigma i}$, while the radius differences between consecutive surfaces are $\Delta r_{\sigma i} \equiv r_{\sigma i+1} - r_{\sigma i}$. The fatigue limit and failure surfaces are defined, respectively, for $i = 1$ and $i = M + 1$, while the remaining $i = 2, 3, \ldots, M$ are the damage surfaces. The damage backstress lengths are always between $|\vec{\beta}'_{\sigma i}| = 0$, if consecutive centers coincide, and $|\vec{\beta}'_i| = \Delta r_{\sigma i}$, if they are mutually tangent.

The proposed IF model uses a 5D damage vector $\vec{D}' \equiv [D_1\ D_2\ D_3\ D_4\ D_5]^T$ that acts as an internal variable that stores the current damage state (to account for the damage memory). The scalars D_1 through D_5 are signed damage quantities associated with each of the directions of the 5D deviatoric stress vector $\vec{s}'$, defined in [10]. The accumulated damage D (analogous to the accumulated plastic strain p) is obtained from the length of the path described by the 5D damage vector $\vec{D}'$, calculated in either continuous or discrete formulations from

$$D = \int dD = \int |d\vec{D}'| \cong \sum \Delta D = \sum |\Delta\vec{D}'| \tag{5}$$

If a given stress state $\vec{s}'$ is on the fatigue limit surface with a normal unit vector $\vec{n}'_\sigma$, and if its infinitesimal increment $d\vec{s}'$ is in the outward direction, then $d\vec{s}'^T \cdot \vec{n}'_\sigma > 0$ and a fatigue damage increment is obtained from a *damage evolution rule* (inspired on the Prandtl-Reuss flow rule [9]):

$$d\vec{D}' = (1/D_\sigma)\cdot(d\vec{s}'^T \cdot \vec{n}'_\sigma)\cdot\vec{n}'_\sigma \cdot f_{MS}(\vec{\sigma})\cdot f_{NP}(\vec{\beta}'_\sigma, \vec{n}'_\sigma) \tag{6}$$

where $f_{MS}(\vec{\sigma})$ is a scalar *mean stress function* of the current 6D stress $\vec{\sigma}$ to account for mean/maximum-stress effects, which can be defined e.g. from Goodman's or Gerber's $\sigma_a\sigma_m$ relations; and $f_{NP}(\vec{\beta}'_\sigma, \vec{n}'_\sigma)$ is a *NP function* to account for the non-proportionality of the load path. For materials that fail due to distributed damage in all directions, the mean stress function $f_{MS}(\vec{\sigma})$ could be based on the current hydrostatic stress σ_h from $\vec{\sigma}$. On the other hand, for materials that fail due to a single dominant crack, like most metals (which require the critical-plane approach), then $f_{MS}(\vec{\sigma})$ could be based on the normal stress $\sigma_\perp$ perpendicular to the considered candidate plane.

Except for the failure surface (which never translates), during this damage process the fatigue limit and all damage surfaces suffer translations

$$d\vec{\beta}'_{\sigma i} = d_{\sigma i}\cdot\vec{v}'_{\sigma i}\cdot dD, \ \text{ if } |\vec{\beta}'_{\sigma i}| < \Delta r_{\sigma i} \quad \text{or} \quad d\vec{\beta}'_{\sigma i} = 0, \ \text{ if } |\vec{\beta}'_{\sigma i}| = \Delta r_{\sigma i} \tag{7}$$

where $d_{\sigma i}$ are coefficients calibrated for each surface, and $\vec{v}'_{\sigma i}$ are the *damage surface translation directions* adapted e.g. from the general translation rule from [9].

The current generalized damage modulus D_σ is then obtained from the consistency condition, which guarantees that the current stress state is never outside the fatigue limit surface, taken from an analogy to the NLK hardening formulation

$$D_\sigma = \left(\sum_{i=1}^{M} d_{\sigma i}\cdot\vec{v}'^{T}_{\sigma i}\right)\cdot\vec{n}'_\sigma \tag{8}$$

allowing the calculation of the evolution of the damage vector $\vec{D}'$ using Eq. 6.

The (scalar) accumulated damage D is then obtained from Eq. 5. This formulation can deal with any multiaxial stress history, proportional or NP, and eliminates the need to count cycles and find equivalent ranges, or even to define them. For instance, Fig. 2 shows IF damage predictions for a material whose elastic Coffin-Manson's parameters are $\sigma_c = 772.5MPa$ and $b = -0.09$, under the uniaxial loading history $\sigma_x = \{0 \rightarrow 300 \rightarrow -300 \rightarrow 300\}MPa$. Jiang-Sehitoglu's translation rule was adopted with $M = 16$ surfaces, calibrated between logarithmically spaced damage levels 10^{-8} and 0.01.

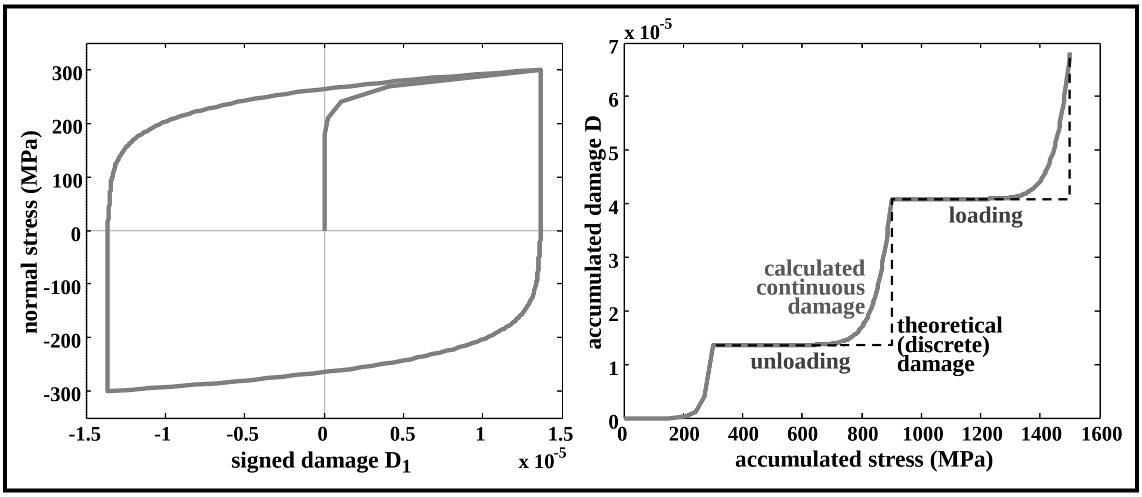

Figure 2. Hysteresis loops relating applied stress and a signed damage state (left) and resulting accumulated damage (right) for a uniaxial constant amplitude loading history.

Strain-Based Incremental Fatigue Formulation

The above example assumed linear elastic histories, with damage calculated from SN models such as Wöhler-Basquin's. But the proposed IF approach can be extended for elastoplastic histories, whose fatigue damage must be quantified by εN models. So, instead of using fatigue limit and damage surfaces defined in stress spaces, strain spaces could be used instead. A generalized damage modulus D_ε (instead of D_σ) is thus defined, which for uniaxial histories becomes the derivative of the normal strain ε with respect to damage D, thus $D_\varepsilon \equiv d\varepsilon/dD$.

In the strain-based version of the proposed IF approach, a 5D deviatoric strain increment $d\vec{e}'$, defined in [8], is used to calculate the associated 5D *damage* increment $d\vec{D}'$ from the current D_ε, using a *damage evolution rule*. *Damage memory* is stored by the current arrangement among *damage surfaces* defined by their *damage backstrains* $\vec{\beta}'_{\varepsilon i}$, from which the *damage surface translation directions* $\vec{v}'_{\varepsilon i}$ are calculated according to some translation rule and combined with material coefficients $d_{\varepsilon i}$ to calculate the current D_ε. The *accumulated damage* D is then equal to the integral of the scalar norm $|d\vec{D}'|$ of the damage increments. The same equations from the stress-based version can be used in the strain-based one, as long as the M damage surface backstrains $\vec{\beta}'_{\varepsilon 1}$, $\vec{\beta}'_{\varepsilon 2}$,

…, $\vec{\beta}'_{\varepsilon M}$, radii $r_{\varepsilon i}$, and radius differences $\Delta r_{\varepsilon i} \equiv r_{\varepsilon i+1} - r_{\varepsilon i}$ between consecutive damage surfaces are all defined as strain (instead of stress) quantities.

RESULTS

The proposed IF formulation is experimentally evaluated using complex 2D tension-torsion stress histories, applied on annealed tubular 316L stainless steel specimens in a multiaxial servo-hydraulic testing machine. The Coffin-Manson curve for this material is $\Delta\varepsilon/2 = 0.0119 \cdot (2N)^{-0.277} + 0.758 \cdot (2N)^{-0.582}$, obtained from uniaxial tests.

The experiments consist of strain-controlled tension-torsion cycles applied to eight tubular specimens, each of them following one of the eight periodic $\varepsilon_x \times \gamma_{xy}/\sqrt{3}$ histories from Figure 3. Table 1 compares the predicted and observed fatigue lives in number of blocks, where each block consists of a full load period. All predictions were performed using the strain-based version of the proposed incremental plasticity formulation, assuming for simplicity $f_{MS}(\vec{\sigma}) \equiv 1$ and $f_{NP}(\vec{\beta}'_{\sigma}, \vec{n}'_{\sigma}) \equiv 1$ in Eq. 6.

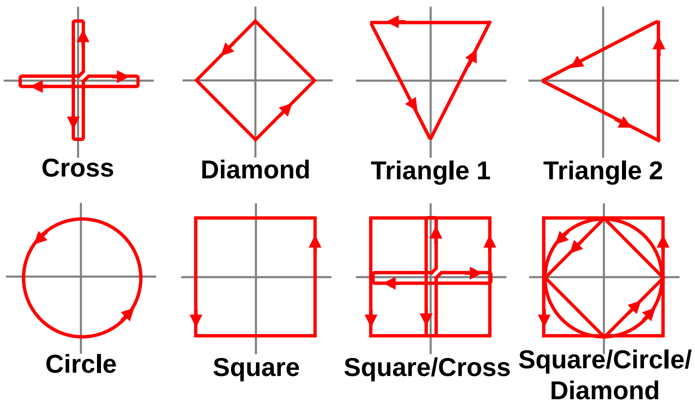

Figure 3. Applied periodic $\varepsilon_x \times \gamma_{xy}/\sqrt{3}$ strain paths on eight tension-torsion tubular specimens, all of them with normal and effective shear amplitudes *0.6%*.

As shown in Table 1, albeit the proposed method does not use any cycle detection or counting algorithm, all fatigue lives were predicted with relatively small errors, within the usual scatter of fatigue measurements. It also automatically applies Miner's rule under VAL, as it can be seen in the loading path consisting of blocks of consecutive square and cross paths, since the predicted number of blocks *482* is such that $1/482 \cong 1/751 + 1/1314$. Similarly, the predicted *327* blocks of consecutive square, circle and diamond paths is such that $1/327 \cong 1/751 + 1/996 + 1/1436$. Miner's rule was also confirmed within the observed experimental results, since e.g. in this latter case it would predict a life of $1/(1/772 + 1/837 + 1/976) = 285$ blocks, almost the same value as the measured *288* blocks. It is important to note that all the predictions were based only on uniaxial Coffin-Manson data, without any posterior curve fitting.

Table 1. Predicted and observed lives, in number of blocks, for each applied path.

Tension-Torsion path:	predicted	observed	error
Cross	1314	1535	–14%
Diamond	1436	976	+47%
Triangle 1	1135	842	+35%
Triangle 2	1180	840	+40%
Circle	996	837	+19%
Square	751	772	–3%
Square + Cross	482	342	+41%
Square + Circle + Diamond	327	288	+14%

CONCLUSIONS

In this work, an Incremental Fatigue formulation was proposed, based on a direct analogy with incremental plasticity models. Both proposed stress and strain-based approaches can be formulated using traditional stress, strain, or even energy-based SN and εN damage models, such as Wöhler-Basquin, Coffin-Manson, Smith-Watson-Topper or Fatemi-Socie, becoming attractive and practical for engineering use. These models did not require additional fitting parameters, or complex calibration routines, as opposed to traditional Continuum Damage Mechanics approaches. The results show that the proposed method is able to predict quite well multiaxial fatigue lives of complex tension-torsion histories, even though it does not require cycle detection, multiaxial rainflow counting, or path-equivalent range computations.

REFERENCES

1. Wang, C.H., Brown, M.W. (1996) *J Eng Mater Technology* **118**, 367-370.
2. Meggiolaro, M.A., Castro, J.T.P. (2012) *Int J Fatigue* **42**, 194-206.
3. Meggiolaro, M.A., Castro, J.T.P. (2012) *Int J Fatigue* **42**, 217-226.
4. Kachanov, L.M. (1986) *Introduction to Continuum Damage Mechanics*, Springer.
5. Wetzel, R.M. (1971) Ph.D. Thesis, U Waterloo.
6. Chu, C.C. (2000) In: ASTM STP 1389, pp. 67-78.
7. Kreiser, D., Jia, S.X., Han, J.J., Dhanasekar, M. (2007) *Int J Fatigue* **29**, 1523-1530.
8. Meggiolaro, M.A., Castro, J.T.P., Wu, H. (2016) *Int J Fatigue* **82**, 158-166.
9. Meggiolaro, M.A., Castro, J.T.P., Wu, H. (2016) *Int J Fatigue* **82**, 167-178.