COMPARISON BETWEEN LINEAR AND NON-LINEAR KINEMATIC HARDENING MODELS TO PREDICT THE MULTIAXIAL BAUSCHINGER EFFECT

M.A. Meggiolaro[1], J.T.P. Castro[1] , H. Wu[2]

[1] Department of Mechanical Engineering, Pontifical Catholic University of Rio de Janeiro
Rua Marquês de São Vicente 225 – Gávea, Rio de Janeiro, RJ, 22453-900, Brazil
[2] School of Aerospace Engineering and Applied Mechanics, Tongji University
Siping Road 1239, 200092, Shanghai, P.R. China

ABSTRACT

The best multiaxial low-cycle fatigue damage models use both the stress and strain histories to quantify fatigue damage accumulation. However, in most engineering applications, either the stress or the strain history is known, but not both. Therefore, to properly reproduce memory effects on stress-strain loops induced by non-proportional (NP) elastoplastic histories, incremental plasticity models are indispensable to correlate infinitesimal changes in all stress components with the associated infinitesimal strain changes, and vice-versa. These models are based on three fundamental equations: (i) the *yield function*, which describes combinations of stresses that lead to plastic flow; (ii) the *plastic flow rule*, which describes the relationship between stresses and plastic strains; and (iii) the *hardening rule*, which defines how the yield resistance changes with plastic straining. To predict the Bauschinger effect under multiaxial conditions, two kinematic hardening formulations using multiple surfaces have been commonly used: the multi-linear Mróz and the non-linear kinematic formulations. In this work, multiaxial stress-strain predictions based on these two formulations are evaluated from elastoplastic strain-controlled tension-torsion experiments performed on tubular specimens of 316L stainless steel, for several challenging NP paths.

KEYWORDS

Multiaxial fatigue; Ratcheting; Incremental plasticity; Non-linear kinematic hardening; Non-proportional loading.

INTRODUCTION

The Bauschinger effect causes kinematic hardening, which can be modeled in stress spaces by allowing the yield surface to translate with no change in its size or shape. The radius *S* of the yield surface is fixed while its center is translated, changing the associated generalized plastic modulus *P* that defines the slope between stress and plastic strain increments in the Prandtl-Reuss plastic flow rule. The main difference among kinematic hardening models is how to obtain the scalar *P* as the yield surface translates, as well as the direction of such translation under general multiaxial loads. Most of these hardening models can be divided into two classes: Mróz multi-surface and non-linear kinematic (NLK) models.

The multi-surface kinematic hardening model proposed by Mróz [1] describes the behavior of elastoplastic solids through a family of nested surfaces in stress space, the innermost being the yield surface associated with the material yield strength. It assumes *P* is piecewise constant, resulting in a multi-linear description of the stress-strain curve. Garud [2] improved this model to avoid intrinsic numerical errors during the surface translation calculations. Despite popular, such multi-linear models cannot predict any uniaxial ratcheting or mean stress relaxation caused by unbalanced loadings, since their idealized uniaxial hysteresis loops are assumed to always perfectly close. In addition, under several non-proportional (NP) loading conditions, these models predict multiaxial ratcheting with a constant rate that never decays, severely overestimating the ratcheting effect measured in practice [3].

Non-linear kinematic (NLK) models introduce non-linearity in the hardening surface translation equations and in the calculation of the modulus P, to better predict the stress-strain history associated with unbalanced loadings. Armstrong and Frederick's single-surface original formulation [4] was improved by Chaboche [5] to include multiple nested surfaces, in a similar framework as the one from Mróz, but with a non-linear instead of multi-linear formulation. In the next section, the multi-surface hardening framework is briefly presented, based on an efficient 5D reduced-order deviatoric stress space.

MULTI-SURFACE HARDENING FRAMEWORK

In the multi-surface framework, the innermost circle is the monotonic yield surface, with radius $r_1 \equiv S_Y$, see Fig. 1. In addition, $M - 1$ *hardening surfaces* with radii $r_1 < r_2 < \ldots < r_{M+1}$ are defined, along with an outermost *failure surface* whose radius r_{M+1} is equal to the true rupture stress σ_U of the material. Their centers are located at points $\vec{s}'_{ci}$ with $i = 2, \ldots, M + 1$, respectively. These nested circles cannot cross one another, must have increasing radii, and for a virgin material they all are initially concentric at the origin of the stress space. The failure surface never translates, i.e. its center always remains at the origin of the stress space, $\vec{s}'_{cM+1} \equiv 0$. Except for the failure surface, all other hardening surfaces can translate as the material plastically deforms and hardens. The difference between the radii of each pair of consecutive surfaces is defined as $\Delta r_i \equiv r_{i+1} - r_i$. In principle, all hardening surfaces radii r_i may change during plastic deformation as a result of isotropic and NP hardening effects. On the other hand, any changes in the stress state fully inside the yield surface are assumed elastic, not resulting in any surface translation as long as $|\vec{s}' - \vec{s}'_{c1}| = r_1$.

The current location of the yield surface center is known as the backstress vector $\vec{\beta}' \equiv \vec{s}'_{c1}$. It can be decomposed as the sum of M surface backstresses $\vec{\beta}'_1, \ldots, \vec{\beta}'_M$ that describe the relative positions $\vec{\beta}'_i \equiv \vec{s}'_{ci} - \vec{s}'_{ci+1}$ between centers of the consecutive hardening surfaces, see Fig. 1. Note that the length (norm) $|\vec{\beta}'_i|$ of each surface backstress is always between $|\vec{\beta}'_i| = 0$, if the surface centers $\vec{s}'_{ci}$ and $\vec{s}'_{ci+1}$ coincide (as in an unhardened condition), and $|\vec{\beta}'_i| = \Delta r_i$, if the surfaces are mutually tangent (a saturation condition with maximum hardening).

All yield and hardening surfaces: (i) must translate as rigid bodies when the point $\vec{s}'$ that defines the current deviatoric stress state in the adopted deviatoric space reaches their boundaries, to guarantee that such stress point is never outside any surface; and (ii) they cannot cross through one another, therefore they gradually become mutually tangent to one another at the current stress point $\vec{s}'$ as the material plastically deforms.

Mróz Linear Formulation

In the Mróz multi-surface formulation, a hardening surface can only translate if the current stress state $\vec{s}'$ reaches its border. The outermost surface that is moving at any instant is called the *active surface*, denoted here as the surface with index i_A. As a result, during plastic straining, all inner surfaces $1, 2, \ldots, i_A - 1$ must be mutually tangent at $\vec{s}'$ (to not cross each other) and translate altogether with the active surface i_A. Thus, their centers do not move relatively to each other, resulting in null increments $d\vec{\beta}_i' = 0$. The Mróz translation rule only needs then to be applied to the evolution $d\vec{\beta}_i'$ of the active surface $i = i_A$, giving

$$d\vec{\beta}_i' = \begin{cases} d\vec{s}_{c_i}' = d\vec{s}_{c_1}' = d\vec{\beta}', & if \ \ i = i_A \\ 0, \ \ if \ \ i \neq i_A \end{cases} \quad (1)$$

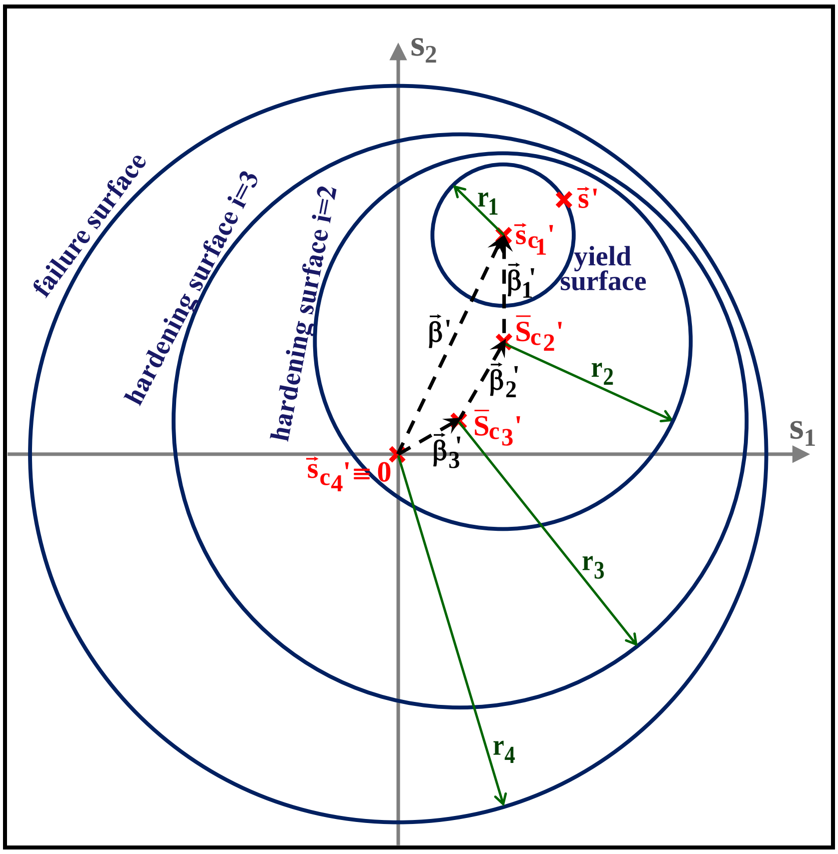

Fig. 1: Yield, hardening, and failure surfaces in the deviatoric space for $M = 3$, showing the backstress vector $\vec{\beta}'$ that defines the location of the yield surface center $\vec{\beta}' \equiv \vec{s}_{c1}'$ and its components $\vec{\beta}_1'$, $\vec{\beta}_2'$, and $\vec{\beta}_3'$ that describe the relative positions between centers.

The kinematic rule for the translation $d\vec{\beta}_i'$ of a surface can be defined from an assumed translation direction $\vec{v}_i'$. Prager [6] assumed that $\vec{v}_i'$ is parallel to the direction of the unit vector $\vec{n}'$ normal to the surface at $\vec{s}'$. Ziegler, on the other hand, assumed $d\vec{\beta}_i'$ happens in the radial direction $\vec{s}' - \vec{s}_{c_i}'$ from the surface center [7]. For von Mises materials, both Prager's and Ziegler's rules result in $\vec{v}_i' \equiv \vec{n}' \cdot (r_{i+1} - r_i) = \vec{n}' \cdot \Delta r_i$. Mróz [1], on the other hand, assumed $d\vec{\beta}_i'$ occurs in a direction $\vec{v}_i' \equiv \vec{n}' \cdot \Delta r_i - \vec{\beta}_i'$, where $-\vec{\beta}_i'$ is called the "dynamic recovery" term.

In the Mróz formulation, each surface is associated with a constant generalized plastic modulus P_i ($i = 1, 2, \ldots, M + 1$), which altogether define a *field of hardening moduli*. The value of P at each instant, to be used in the Prandtl-Reuss plastic flow rule, is then chosen as the P_i from the active surface $i = i_A$. Such piecewise-constant values of P result in a multi-linear representation of the stress-strain curve, in a so-called "uncoupled formulation" [8].

The Mróz formulation results in very good plasticity predictions for proportional problems without significant mean stresses or ratcheting. However, the directions of the calculated stress paths may significantly vary depending on the number of surfaces used, while better predictions are not necessarily obtained from using a larger number of surfaces. Moreover, it is not able to correctly predict ratcheting and mean stress relaxation effects, because of its multi-linear (instead of non-linear) representation of the stress-strain behavior. A more critical problem happens e.g. for a stress state contouring a hardening surface, where the Mróz formulation wrongfully predicts zero plastic straining even though the yield surface is clearly moving in circles. A better approach is to replace such multi-linear models with a non-linear kinematic hardening formulation, described next.

Non-Linear Kinematic (NLK) Formulation

During plastic straining, in the NLK multi-surface formulation [4-5] *P* is assumed coupled with the directions $\vec{v}_i'$, while the yield and *all* hardening surfaces are translated, as opposed to the Mróz formulation, where all surfaces outside the active one would not move. Thus, plastic straining causes non-zero increments $d\vec{\beta}_i' = p_i \cdot \vec{v}_i' \cdot dp$ for the yield ($i = 1$) and all non-saturated hardening surfaces $i = 2, \ldots, M$, where dp is the equivalent plastic strain increment and p_i is a generalized plastic modulus coefficient that must be calibrated for every hardening surface i, used in the calculation of *P*. The main difference among the several NLK hardening models proposed in the literature rests in the equation for $\vec{v}_i'$, which for most models can be condensed into the general equation [9]

$$\vec{v}_i' = \vec{n}' \cdot \Delta r_i - \chi_i{*} \cdot m_i{*} \cdot \gamma_i \cdot [\, \delta_i \cdot \vec{\beta}_i' + (1-\delta_i) \cdot (\vec{\beta}_i'^T \cdot \vec{n}') \cdot \vec{n}' \,] \tag{2}$$

$$\chi_i{*} \equiv \left(\frac{|\vec{\beta}_i'|}{\Delta r_i} \right)^{\chi_i} \quad \text{and} \quad m_i{*} \equiv \begin{cases} \left[\vec{\beta}_i'^T \cdot \vec{n}' / |\vec{\beta}_i'| \right]^{m_i}, & if \;\; \vec{\beta}_i'^T \cdot \vec{n}' \geq 0 \\ 0, \;\; if \;\; \vec{\beta}_i'^T \cdot \vec{n}' < 0 \end{cases} \tag{3}$$

which includes calibration parameters known as the ratcheting exponent χ_i, the multiaxial ratcheting exponent m_i, the ratcheting coefficient γ_i, and the multiaxial ratcheting coefficient δ_i. Table 1 compares the main differences between the Mróz and NLK formulations.

	Mróz multi-surface	*NLK multi-surface*
Surface translation during plastic straining:	no translation outside active surface	yield and all hardening surfaces (including bounding surface) translate
Surface translation direction $\vec{v}_i'$:	defined by linear rules such as Mróz	defined by non-linear rules
Variation $d\vec{\beta}_1'$ during plastic straining:	$d\vec{\beta}_i' \neq 0$ only for the active surface $i = i_A$	all backstress increments $d\vec{\beta}_1'$ ($i = 1, 2, \ldots, M$) can be different than zero
Generalized plastic modulus *P*:	piecewise-constant $P = P_i$ from the active surface $i = i_A$	non-linear and continuously varying, calculated from relative positions among all surfaces
Consistency condition that prevents $\vec{s}'$ from moving outside any surface:	used to calculate $d\vec{\beta}_1'$ and associated translations $d\vec{s}c_i'$	used to calculate the non-linear value of *P*

Table 1: Comparison between the Mróz and NLK multi-surface model formulations to predict multiaxial kinematic hardening effects.

EXPERIMENTAL VALIDATION

Both Mróz and NLK formulations were computationally implemented, to compare their prediction potential under multiaxial NP conditions. Isotropic, NP, and several versions of the Mróz/Garud and NLK models were simulated for various representative load paths. To improve the calculation accuracy, the backstress was divided into 10 additive components, resulting in $M = 10$ yield and hardening surfaces, adopted in all simulations for a fair comparison.

Tension-torsion experiments were then performed on tubular annealed 316L stainless steel specimens in a multiaxial testing machine. Engineering stresses and strains were measured using a load/torque cell and an axial/torsional extensometer. The cyclic properties of this steel were obtained from uniaxial tests, which were then used to calibrate the parameters of all simulated models, to be used in the predictions of the multiaxial NP behavior. Figure 2 shows the applied strain-controlled histories for six strain paths $\varepsilon_x \times \gamma_{xy}/\sqrt{3}$, as well as the predicted and measured stress paths after isotropic and NP hardening stabilization, using Jiang-Sehitoglu's [10] NLK model. A relatively good agreement has been found using NLK models, while similar simulations using Mróz/Garud's multi-linear models failed to converge to stabilized loops, wrongfully predicting a net plastic strain accumulation (false ratcheting).

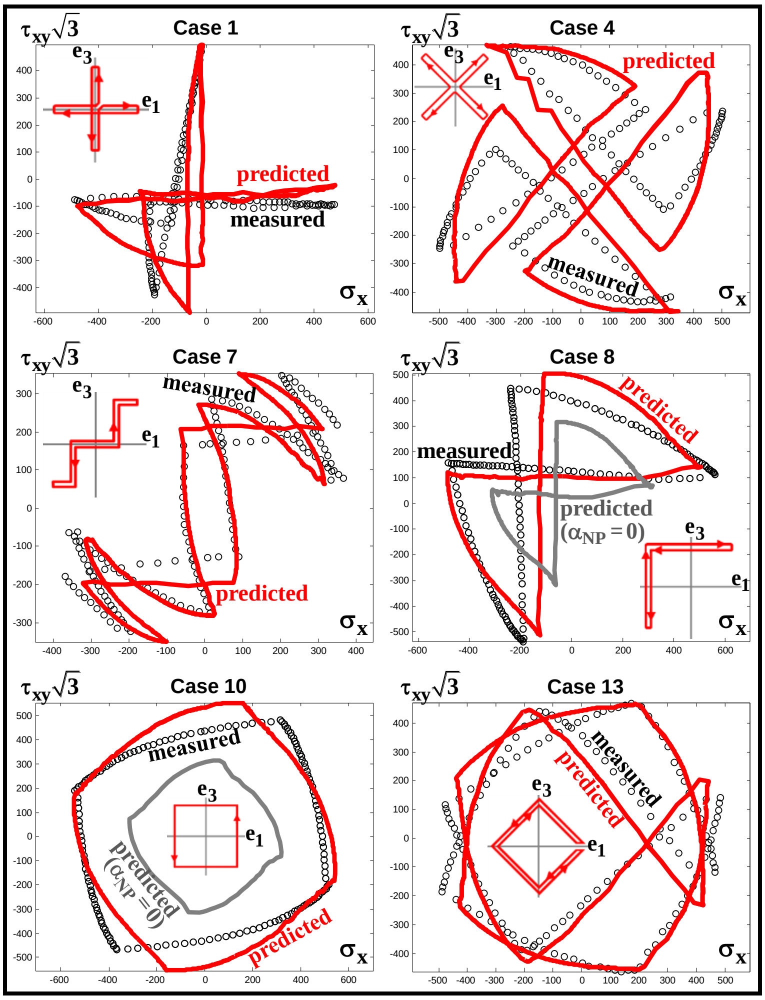

Fig. 2: Experimental and *predicted* stress vs. shear stress paths (in MPa) using NLK models, for several strain-controlled tension-torsion histories.

CONCLUSIONS

Multiaxial stress-strain predictions based on Mróz and NLK formulations were evaluated from elastoplastic strain-controlled tension-torsion experiments on 316L stainless steel tubular specimens, for several challenging NP paths. It was found that the Mróz formulation has several issues with NP loadings, wrongfully predicting ratcheting even in balanced loadings, or zero plastic straining in elastoplastic circular paths. The NLK formulation, on the other hand, can deal with unbalanced loadings, being able as well to predict uniaxial and multiaxial ratcheting and mean stress relaxation. The NLK formulation is thus strongly recommended over the popular Mróz/Garud approach for cyclic variable-amplitude loadings, ultimately resulting in better low-cycle fatigue life predictions under multiaxial loads.

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Corresponding author: meggi@puc-rio.br