# EVALUATION OF THE ERRORS INDUCED BY HIGH NOMINAL STRESSES IN THE CLASSICAL εN METHOD

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The traditional εN procedures are inconsistent when modeling nominal stresses and strains by Hooke's law and the stresses and strains at the critical notch root by Ramberg-Osgood's equation, since the material is the same at both regions. Moreover, when the nominal stresses are not substantially smaller than the cyclic yielding strength $\mathbf{S_{Yc}}$, the hysteresis loops at the notch root predicted by such classical Neuber approach can be significantly non-conservative. To avoid this serious problem, it is mandatory to use Ramberg-Osgood to model both the nominal and the critical stresses and strains. In this work, the errors induced by the Hookean modeling of the nominal stresses are evaluated from a comprehensive study on measured properties of 517 structural steels. An effective procedure for the numerical solution of Neuber's system when describing the nominal stresses by Ramberg-Osgood is introduced. This methodology was required to solve an important residual life problem, which involved a potentially catastrophic environmental hazard.

## *INTRODUCTION*

The εN is a modern fatigue design method (Dowling [1], Fuchs and Stephens [2], Rice [3], Sandor [4], Castro and Meggiolaro [5]) in which Neuber is the most used equation to correlate the nominal stress $\Delta\sigma_n$ and strain $\Delta\varepsilon_n$ ranges with the stress $\Delta\sigma$ and strain $\Delta\varepsilon$ ranges they induce at a notch root. The Neuber equation states that the product between the stress concentration factor $\mathbf{K_\sigma}$ (defined as $\Delta\sigma/\Delta\sigma_n$) and the strain concentration factor $\mathbf{K_\varepsilon}$ (defined as $\Delta\varepsilon/\Delta\varepsilon_n$) is constant and equal to the square of the geometric stress concentration factor $\mathbf{K_t}$, thus

$$\mathbf{K_t^2} = \frac{\Delta\sigma \cdot \Delta\varepsilon}{\Delta\sigma_n \cdot \Delta\varepsilon_n} \qquad (1)$$

Some authors prefer to use $\mathbf{K_f}$, the fatigue concentration factor, instead of $\mathbf{K_t}$ in this equation (Topper [6]). When the nominal stresses are lower than $\mathbf{S_{Yc}}$, the cyclic yielding strength, it is common practice to model them as Hookean and, therefore,

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to use the Neuber equation in the simplified form

$$\mathbf{K_t^2} = \frac{\Delta\sigma \cdot \Delta\varepsilon \cdot \mathbf{E}}{\Delta\sigma_n^2} \qquad (2)$$

Ramberg-Osgood is one of many empirical relations that can be used to model the cyclic response of the materials. Its main limitation is not to recognize a purely elastic behavior, and its main advantage is its mathematical simplicity. It can be used to describe the stresses and strains at the notch root by

$$\Delta\varepsilon = \frac{\Delta\sigma}{E} + 2\left(\frac{\Delta\sigma}{2H_c}\right)^{1/hc} \qquad (3)$$

where $\mathbf{E}$ is the Young's modulus, $\mathbf{H_c}$ is the hardening coefficient and $\mathbf{h_c}$ is the hardening exponent of the cyclically stabilized $\Delta\sigma\Delta\varepsilon$ curve.

Eliminating $\Delta\varepsilon$ from Equations (2) and (3), $\Delta\sigma_n$ is directly related to $\Delta\sigma$ by

$$K_t^2\,\Delta\sigma_n^2 = \Delta\sigma^2 + \frac{2E\Delta\sigma^{(h_c+1)/h_c}}{(2H_c)^{1/h_c}} \qquad (4)$$

However, the above equation is logically incongruent, since it treats the same material by two different models: Ramberg-Osgood at the notch root and Hooke at the nominal region. Moreover, this procedure can generate significant numerical errors even when the nominal stresses are much lower than the material cyclic yielding strength, as it will be discussed next.

## *LIMITATIONS OF THE CLASSICAL NEUBER APPROACH*

Consider for instance a piece made of hot-rolled SAE 1009 steel, with $\mathbf{E}$ = 207GPa, $\mathbf{h_c}$ = 0.12, $\mathbf{H_c}$ = 462MPa, $\mathbf{S_{Yc}}$ = 219MPa (Rice [3]). Let's calculate the stress at a notch root with a stress concentration $\mathbf{K_t}$ = 1.3, associated with a nominal stress amplitude $\Delta\sigma_n/2$ of 200MPa. Since this nominal stress amplitude is smaller than $\mathbf{S_{Yc}}$, the classical $\varepsilon\mathbf{N}$ methodology assumes that Equation (4) applies, which results in a notch-root stress amplitude $\Delta\sigma/2 \cong$ 195MPa, smaller than the nominal stress 200MPa. This result is a clear non-sense, since the notch root stresses must always be greater than the nominal ones (in modulus). To avoid these type of errors induced by the classical Neuber approach, it is necessary to use the Ramberg-Osgood model to describe not only the stresses at the notch root, but also to describe the nominal stresses, writing

$$\frac{\Delta\varepsilon_n}{2} = \frac{\Delta\sigma_n}{2E} + \left(\frac{\Delta\sigma_n}{2H_c}\right)^{1/h_c} \qquad (5)$$

In this case, given $\Delta\sigma_n$, the stress range at the notch root $\Delta\sigma$ can be calculated from Equations (1), (3), and (5), giving

$$K_t^2\left(\Delta\sigma_n^2+\frac{2E\Delta\sigma_n^{(h_c+1)/h_c}}{(2H_c)^{1/h_c}}\right)=\Delta\sigma^2+\frac{2E\Delta\sigma^{(h_c+1)/h_c}}{(2H_c)^{1/h_c}} \qquad (6)$$

If Equation (6) is applied to the SAE 1009 example discussed before, then the stress amplitude at the notch root can be calculated as $\Delta\sigma/2 \cong 218$MPa, a much more reasonable value for a nominal amplitude of 200MPa with $K_t = 1.3$.

Figure 1 shows a comparison between the stress $K_\sigma$ and strain $K_\varepsilon$ concentration factors predictions made by the classical Neuber approach using Equation (4), and the general (corrected) ones obtained using Equation (6), for the SAE 1009 steel when the notch root has a $K_t$ of 3.

As it can be seen in the figure, for nominal stress amplitudes $\Delta\sigma_n/2$ smaller than $0.5{\cdot}S_{Yc}$ both predictions result in roughly the same concentration factors. However, for larger nominal stress values the predictions diverge, and the classical Neuber approach wrongfully predicts ever increasing strain concentration factors $K_\varepsilon$ and even stress concentration factors $K_\sigma$ smaller than unity.

Note also that the general Neuber formulation implies that both $K_\sigma$ and $K_\varepsilon$ tend to a constant value as the nominal stress amplitude is increased. According to Neuber's equation, any material that follows Ramberg-Osgood's equation presents this same behavior. These constant values can be calculated from Equation (6), assuming that the elastic component of both nominal and notch-root strains are negligible compared to the respective plastic strain components, resulting in

$$K_t^2\left(\frac{2E\Delta\sigma_n^{(h_c+1)/h_c}}{(2H_c)^{1/h_c}}\right)=\frac{2E\Delta\sigma^{(h_c+1)/h_c}}{(2H_c)^{1/h_c}} \Rightarrow K_\sigma=\frac{\Delta\sigma}{\Delta\sigma_n}=K_t^{2h_c/(1+h_c)} \qquad (7)$$

From Equation (7) and using that $K_\sigma{\cdot}K_\varepsilon = K_t^2$, then lower and upper bounds can be calculated for $K_\sigma$ and $K_\varepsilon$

$$K_t^{2h_c/(1+h_c)} \le K_\sigma \le K_t \le K_\varepsilon \le K_t^{2/(1+h_c)} \qquad (8)$$

In addition, it is found that the errors in $\Delta\sigma$ are not a strong function of $K_t$, being mainly dependent on the nominal stress range $\Delta\sigma_n$. These errors tend to slightly decrease as $K_t$ is increased, reaching a constant value for very high stress concentration factors. Therefore, the results presented in this work for $K_t$ equal to 3 can be extended to any stress concentration factor.

To quantitatively account for the errors induced by the Hookean modeling of the nominal stresses, a detailed study has been performed on measured properties of 517 different structural steels. Figure 2 shows the probability density functions (pdf) of the percentage error in the stress ranges calculated by the classical Neuber approach for these 517 steels, considering $\mathbf{K_t} = 3$.

As it can be seen in the figure, the Hookean modeling can lead to errors as high as 50% in the calculated stress at the notch root, especially if the nominal stress amplitude $\mathbf{\Delta\sigma_n}/2$ is much above the cyclic yielding strength $\mathbf{S_{Yc}}$. However, even if the nominal stresses are much smaller than $\mathbf{S_{Yc}}$, the errors induced by the classical Neuber approach are very significant, reaching values up to 23% in some cases. And due to the non-linearities of the Coffin-Manson curve, these errors in stress translate to much higher non-conservative errors in life prediction. To visualize this, Figure 3 shows the probability density functions of the errors in the lives predicted by the classical Neuber approach, calculated from measured Coffin-Manson data of those 517 structural steels.

As it can be seen in Figure 3, inadmissible non-conservative life prediction errors can be generated using Equation (4). In addition, significant non-conservative errors may be present for virtually any nominal stress amplitude, even for those well under $\mathbf{S_{Yc}}$. For instance, nominal stress amplitudes of only $0.3{\cdot}\mathbf{S_{Yc}}$ can lead in some materials to errors higher than 100% in life prediction, while values close to $\mathbf{S_{Yc}}$ may result in errors up to 2,000%. Depending on the considered material, even nominal stresses as low as $0.1{\cdot}\mathbf{S_{Yc}}$ can result in significant non-conservative errors. In summary, it is mandatory to use Ramberg-Osgood to model both the nominal and the critical stresses and strains, as shown in Equation (6), otherwise completely wrong life predictions may be obtained.

However, the numerical solution of the general Neuber system (considering elastic-plastic nominal stresses) is not trivial to implement. The next section presents the methodology required to warrant correct numerical predictions of the critical loops considering elastic-plastic nominal stresses.

## *NUMERICAL SOLUTION OF THE NEUBER SYSTEM*

To solve Equation (6), a numerical method has been developed based on the fact that this equation essentially constitutes the combination of two straight lines when represented in bi-logarithmic scale. Since the Newton-Raphson method is very efficient to solve equations with approximately constant derivative, it was adapted to the bi-logarithmic scale. The procedure for the solution of Equation (6) can be summarized by:

- finding the initial guess $\mathbf{\Delta\sigma_0}$ for the value of the notch-root stress range $\mathbf{\Delta\sigma}$

$$\mathbf{\Delta\sigma_0 = \min\{\sqrt{\delta},\ (\delta/\gamma)^{h_c/(1+h_c)}\}} \qquad (9)$$

where the **min** function returns the smaller between two values, and

$$\delta = K_t^2(\Delta\sigma_n^2 + \frac{2E\Delta\sigma_n^{(h_c+1)/h_c}}{(2H_c)^{1/h_c}}),\ \gamma = \frac{2E}{(2H_c)^{1/h_c}} \qquad (10)$$

Equation (9) evaluates if the notch-root stress range is in the predominantly elastic or plastic region, taking as initial value the closest one to the solution.

- calculating the value of $\Delta\sigma_{i+1}$ of the next iteration as a function of the $\Delta\sigma_i$ value

$$\Delta\sigma_{i+1} = \Delta\sigma_i \cdot \left[\frac{\Delta\sigma_i{}^2 + \gamma\cdot\Delta\sigma_i{}^{1+1/h_c}}{\delta}\right]^{-\left(\frac{\Delta\sigma_i{}^2+\gamma\cdot\Delta\sigma_i{}^{1+1/h_c}}{2\Delta\sigma_i{}^2+\gamma(1+1/h_c)\Delta\sigma_i{}^{1+1/h_c}}\right)} \qquad (11)$$

- defining $(\xi-1)$ as the maximum relative error, the iterations proceed until

$$\beta\cdot(\xi\cdot\Delta\sigma_i)^2 + \gamma\cdot(\xi\cdot\Delta\sigma_i)^{1+1/h_c} < \delta \qquad (12)$$

It was found that in most cases only 3 iterations are necessary considering a precision of 0.1% ($\xi$ = 1.001), compared to over 15 iterations from the traditional Newton-Raphson method.

In summary, the numerical procedure shown in this section is able to solve Neuber fast and accurately considering elastic-plastic nominal stresses. This enabling methodology presented in this work has been successfully implemented in a general-purpose fatigue design program named **ViDa**, developed to automate the fatigue design routines by all local methods (Meggiolaro and Castro [7]).

## *APPLICATION TO THERMAL FATIGUE*

The presented study was used to predict the remaining life of chemical ducts under thermal fatigue. The ducts are made of steel with ultimate strength $\mathbf{S_U}$ = 415MPa, yielding strength $\mathbf{S_Y}$ = 227MPa, Young's modulus $\mathbf{E}$ = 206GPa, hardening properties $\mathbf{h_c}$=0.24 and $\mathbf{H_c}$=1058MPa, and cyclic yielding strength $\mathbf{S_{Yc}}$ = 238MPa.

The operating temperatures vary between 60 and 80$^o$C, working under internal pressures between 6 and 15kg/cm$^2$. A Finite-Element analysis of the duct was performed considering both temperature and pressure effects, but neglecting any surface defect. It was found that the thermal bending stresses were about 8 times higher than the stresses induced by the internal pressure, therefore the temperature history was considered as the main input in the fatigue calculations. The Mises stresses at the critical point associated with the temperatures 60 and 80$^o$C were calculated as 177 and 250MPa respectively, from the FE analysis considering the maximum internal pressure of 15kg/cm$^2$.

These stresses were considered as nominal ones, since they didn't account for small defects on the duct surface, such as corrosion pits. A stress concentration factor $\mathbf{K_t} = 3$ was then arbitrated in the predictions, which is of the order of the concentration factors generated by semi-spherical surface cavities in solids (Timoshenko and Goodier [8]).

Under such elastic-plastic stresses it is necessary to apply the εN methodology (instead of the SN) to predict the fatigue crack initiation life of the equipment. Furthermore, the classical approximation of Hookean nominal stresses induces in this case non-conservative errors of over 100% in the predicted lives, because the minimum and maximum notch-root stresses are underestimated as 275 and 321MPa (instead of the expected values of 309 and 405MPa from the general formulation). These predictions, calculated using the **ViDa** software (Meggiolaro and Castro [7]), showed the importance of the general formulation of the Neuber system even for nominal stresses significantly below the cyclic yielding strength $\mathbf{S_{Yc}}$.

## CONCLUSIONS

This paper studied some inconsistencies in the traditional εN procedures, in particular when modeling nominal stresses by Hooke's law and the stresses and strains at the critical notch root by Ramberg-Osgood's equation. From a study on measured properties of 517 structural steels, it was found that high non-conservative life prediction errors can be obtained if the nominal stresses are modeled as purely elastic, even if such stresses are significantly below the material cyclic yielding strength. An effective procedure for the numerical solution of the general formulation of the Neuber equation has been introduced and implemented on a general-purpose fatigue design program. This extended methodology was required to solve an important residual life problem, since the classical approach would induce non-conservative life prediction errors of over 100%.

## REFERENCE LIST

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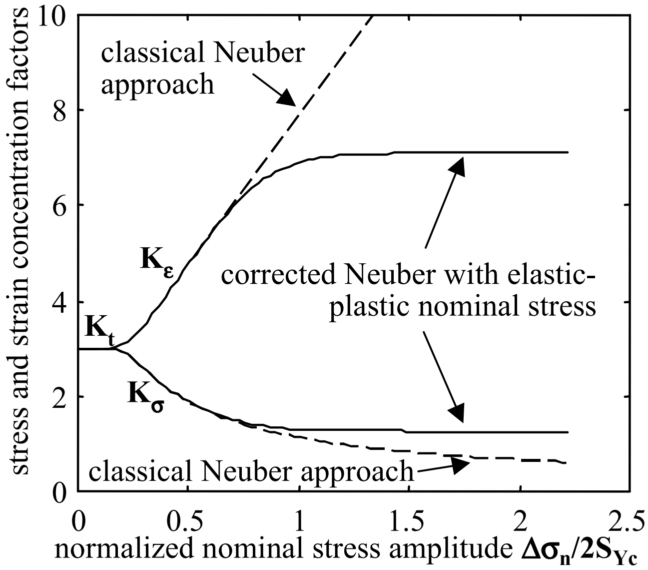


FIGURE 1 Stress and strain concentration factors calculated by both Neuber approaches (SAE 1009 steel, $\mathbf{K_t}$ = 3).

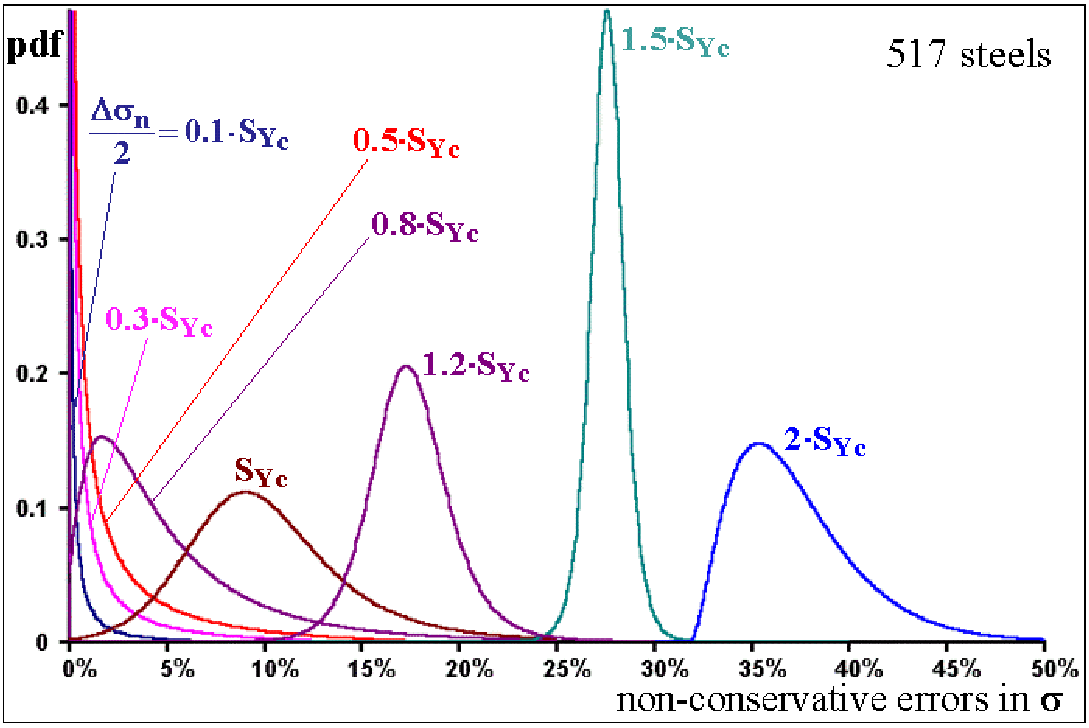


FIGURE 2 Statistics of the errors in notch-root stress predicted by the classical Neuber approach, for several nominal stress levels (517 steels, $\mathbf{K_t}$ = 3).

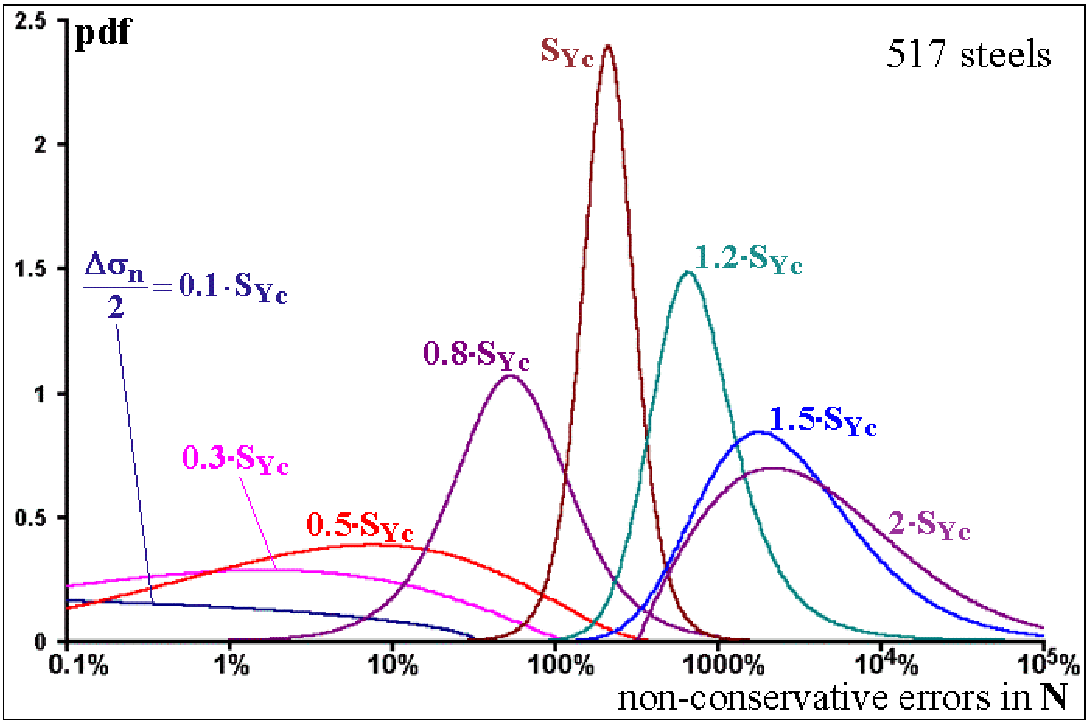


FIGURE 3 Statistics of the non-conservative errors in lives predicted by the classical Neuber approach, for several nominal stress levels (517 steels, $\mathbf{K_t}$ = 3).