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Elastoplastic nominal stress effects in the estimation of the notch-tip behavior in tension

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ABSTRACT

The evaluation of notch effects in low-cycle fatigue problems in general requires local plasticity calculations via the solution of a global Finite Element (FE) problem to obtain the cyclic elastoplastic (EP) stresses and strains at the notch tip, as well as the stress gradient effects around it. Moreover, this EP global FE approach needs to adopt an incremental plasticity formulation in every element of the studied piece that suffers plastic strains. Besides being not trivial to implement, these calculations are computationally intensive, especially when dealing with long loading histories, since they imply in having to solve the EP FE problem for the entire piece for every load increment of every load cycle. A much simpler approach is to perform a single linear elastic (LE) FE calculation on the entire piece for a static unit value of each applied loading, to find the stress and strain *influence factors*. The resulting LE values then require EP corrections to reproduce the true stresses and strains at the critical point of the component. Thus, an EP strain concentration rule must be assumed to estimate the actual notch-tip stresses and strains from the LE values. Perhaps the most used concentration rules are the ones proposed by Neuber and Glinka, which usually result in reasonable estimates in tension, especially under plane-stress conditions. However, most implementations of Neuber's and Glinka's rules assume the nominal stresses (which act away from the notch tip) are purely elastic, which can induce significant numeric errors even at stress levels much below the yield strength. In this work, Neuber's and Glinka's rules are presented in a formulation that assumes nominal stresses as EP instead of LE, highly improving the EP notch corrections, even under gross yielding of the net section. EP FE simulations on thin and thick specimens are used to verify the effectiveness of the proposed formulation, as well as to study 3D notch effects.
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1. Introduction

Linear elastic (LE) concentration factors K_t can be directly used in stress analyses to multiply nominal stresses if the notch tip remains elastic, but not if it yields under those loads. As the notch yields, the elastoplastic (EP) stress concentration factor $K_\sigma \equiv \sigma/\sigma_n$ tends to become lower while the corresponding EP strain concentration factor $K_\varepsilon \equiv \varepsilon/\varepsilon_n$ tends to become higher than K_t (maybe except if the notch-tip radius is much smaller than the piece thickness, when the linear strain concentration rule [1] would predict up to $K_\varepsilon \cong K_t$).

Moreover, strain-gages hardly ever can be bonded at notch tips, due to access limitations. Even modern optical methods such as digital image correlation are still not practical to measure service strain histories at notch tips. In fact, these histories can almost never be directly measured. They must instead be *calculated* from measured nominal strain histories using suitable strain concentration rules, considering the cyclic hardening behavior of the material at the notch tip.

Cyclic softening can present some modeling challenges in both notched and un-notched pieces. Indeed, fatigue tests can become unstable under load control if the material cyclically softens [2]. Since structural components usually work under imposed loads, they may also face this problem. However, in most practical components, cyclic yielding occurs only on the critical notch neighborhood, which remains enclosed by elastic regions under service loads. Hence, the small amount of material around their critical notch tips that suffers EP fatigue damage and softens usually is contained by the rest of the component, which works under LE conditions and forces the notch tip to work under imposed $\Delta\varepsilon$ conditions.

In other words, as fatigue cracks nearly always start from notch tips, the size of the region that cyclically yields usually is small compared to structural component sizes. In these cases, the region that suffers EP fatigue damage is contained within a much larger

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dominant LE field that surrounds and controls it, to maintain the required strain compatibility in the whole component. This is maybe the best justification to prefer εN over SN tests (when testing small fatigue specimens).

It is thus a good practice to force strain histories $\Delta\varepsilon(t)$ on εN specimens in order to check structural component life predictions. However, to be reliable, these $\Delta\varepsilon(t)$ histories must be *identical* to the strain histories that load their critical notch tips in service. In fact, it is a truism to say that to verify εN predictions, $\Delta\sigma\Delta\varepsilon$ loop histories induced by service loads at critical points must be fully reproduced in such εN tests. Despite evident, this warning must nevertheless be emphasized, because to measure strains at critical points is usually a difficult task, as discussed above. So, the usual practice is to measure or to estimate the stress/strain histories induced by the loading in convenient *nominal* points, to then use them to *calculate* the *critical strain histories* using a strain concentration rule, as schematized in Fig. 1. However, imprecise calculations obviously cannot be associated with reliable fatigue life predictions. That is the reason why it is so important to apply accurate strain concentration rules in practical design applications.

For example, Neuber's strain concentration rule [3] is usually adopted assuming the nominal stresses are LE. But nominal stress amplitudes may be associated with nominal plastic strains even if they are lower than the cyclic yield strength S_{Yc}, as long as they are above the elastic limit σ_E of the material, which usually is much smaller than the monotonic S_Y or the cyclic S_{Yc}. Neglecting such nominal plastic strains, i.e. assuming the nominal stresses follow Hooke's law and not e.g. Ramberg–Osgood, may result in large errors in Neuber's predictions.

Such errors are not algebraic, since they occur even if the calculations are numerically correct. In fact, such errors are of a completely different kind, since they are due to a classic εN hypothesis limitation. Indeed, even though the assumption "$\Delta\sigma_n$ is elastic" is traditionally used in most εN calculations, it does *not* correctly reproduce the physics of the problem, unless σ_n is below not only S_Y or S_{Yc} but also the much lower elastic limit σ_E of the material. To avoid this conceptual (and numeric) error, it is necessary to recognize that $\Delta\sigma_n$ is instead EP, as studied next.

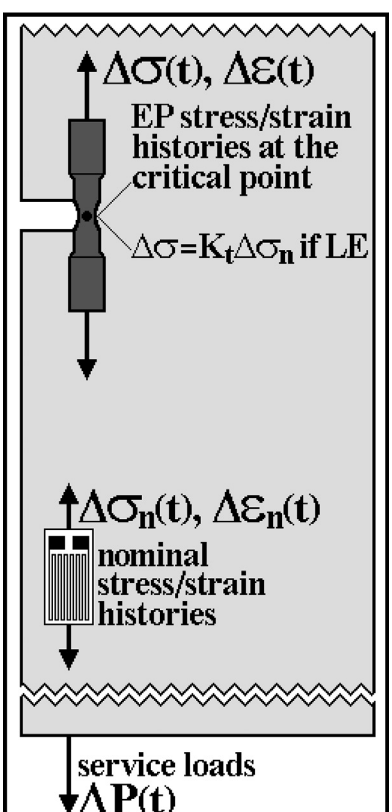

Fig. 1. Strain histories at the critical point must usually be *calculated* from the nominal stress/strain histories using a strain concentration rule.

2. Neuber's strain concentration rule with elastoplastic nominal stresses

To start with, writing Neuber's rule as $(K_t \cdot \Delta\sigma_n)^2 = \Delta\sigma \cdot \Delta\varepsilon \cdot E$ is an inconsistent practice even when $\Delta\sigma_n$ is elastic, because this formulation uses two different equations to describe the same material. Indeed, the usual practice of modeling $\Delta\sigma_n$ as LE to calculate the nominal strain range $\Delta\varepsilon_n$ by Hooke's law, while at the same time using Ramberg–Osgood (which does not recognize purely elastic strains) to describe $\Delta\sigma\Delta\varepsilon$ loops at the notch tip, is at least inelegant even when $\Delta\sigma_n \ll 2S_Y$.

Moreover, when the nominal stress $\Delta\sigma_n$ is EP, Neuber's rule clearly cannot be used in the simplified form $(K_t \cdot \Delta\sigma_n)^2 = \Delta\sigma \cdot \Delta\varepsilon \cdot E$. These problems must be modeled by a system of three equations containing Neuber and two Ramberg–Osgood equations, one to describe the $\Delta\sigma\Delta\varepsilon$ loops at the notch tip, and the other to model the nominal $\Delta\sigma_n\Delta\varepsilon_n$ loops:

$$\begin{cases} K_t^2 = \Delta\sigma\Delta\varepsilon/\Delta\sigma_n\Delta\varepsilon_n \\ \Delta\varepsilon = (\Delta\sigma/E) + 2(\Delta\sigma/2H_c)^{1/h_c} \\ \Delta\varepsilon_n = (\Delta\sigma_n/E) + 2(\Delta\sigma_n/2H_c)^{1/h_c} \end{cases} \tag{1}$$

where H_c and h_c are Ramberg–Osgood's cyclic hardening coefficient and exponent, and E is Young's modulus. Note that all stresses and strains used in this work are true, not engineering values.

Contrary to what might be anticipated, this three-equation system does not complicate too much fatigue life calculations. In fact, the calculation technique is identical to the traditional one: first obtain the notch-tip stress range $\Delta\sigma$ from the nominal EP ranges $\Delta\sigma_n$ and $\Delta\varepsilon_n$, using Neuber and Ramberg–Osgood; then the corresponding $\Delta\varepsilon$ range from Ramberg–Osgood again; and finally the life N from Coffin–Manson or other suitable εN rule:

$$K_t^2\left[\Delta\sigma_n^2 + 2E\Delta\sigma_n\left(\frac{\Delta\sigma_n}{2H_c}\right)^{1/h_c}\right] = \Delta\sigma^2 + 2E\Delta\sigma\left(\frac{\Delta\sigma}{2H_c}\right)^{1/h_c}$$
$$\Rightarrow \Delta\varepsilon = \frac{\Delta\sigma}{E} + 2\cdot\left[\frac{\Delta\sigma}{2H_c}\right]^{1/h_c} = \frac{2\sigma_c}{E}(2N)^b + 2\varepsilon_c(2N)^c \tag{2}$$

To use Neuber's system with the cyclic $\sigma\varepsilon$ (instead of the hysteresis loop $\Delta\sigma\Delta\varepsilon$) curve, simply drop the Δ and the 2 factor in Eq. (2) to obtain

$$K_t^2\left[\sigma_n^2 + E\sigma_n(\sigma_n/H_c)^{1/h_c}\right] = \sigma^2 + E\sigma(\sigma/H_c)^{1/h_c} \tag{3}$$

For instance, let's use Neuber to estimate the life of a piece made from a 1015 steel, knowing it has a notch with geometric LE $K_t = 2$ and is loaded by $\Delta\sigma_n = 500$ MPa. Substituting the 1015 mechanical properties into Eq. (2):

$$2^2\left[500^2 + \frac{2\cdot 207,000}{(2\cdot 945)^{1/0.22}}500^{1.22/0.22}\right]$$
$$= \Delta\sigma^2 + 5.30\cdot 10^{-10}\Delta\sigma^{5.55} \Rightarrow \Delta\sigma = 671 \quad \therefore \Delta\varepsilon = 2.13\%$$
$$= \frac{2\cdot 827}{207,000}(2N)^{-0.11} + 2\cdot 0.95\cdot(2N)^{-0.64} \quad \therefore N$$
$$= 743 \text{ cycles} \tag{4}$$

Notice that since $\Delta\sigma_n = 500 > 2 \cdot S_{Yc} = 482$ MPa, the EP Eq. (2) is mandatory in this problem. In fact, if $\Delta\sigma_n \ll 2 \cdot S_{Yc}$, it is (numerically) irrelevant to work with elastic or elastoplastic models, but the difference between fatigue life predictions based on LE or EP nominal stresses can be very large for higher load ranges. For instance, let's now compare the LE and EP fatigue life predictions for the same 1015 steel piece above, considering:

1. $\Delta\sigma_n = 100$ MPa $\ll 2 \cdot S_{Yc}$: $(2 \cdot 100)^2 = \Delta\sigma_{LE}^2 + 5.3 \cdot 10^{-10}\Delta\sigma_{LE}^{5.55}$, thus $\Delta\sigma_{LE} = 193.5$ MPa, $\Delta\varepsilon_{LE} = 0.0966\%$ and $N_{LE} = 1.2 \cdot 10^8$ cycles (the life would be infinite by SN procedures), while by Eq. (2) $\Delta\sigma = 194.1$ MPa, $\Delta\varepsilon_{EP} = 0.0970\%$ and $N_{EP} = 1.15 \cdot 10^8$ cycles. Hence, for such a low nominal stress, the difference between the LE and EP predictions is irrelevant.
2. $\Delta\sigma_n = 480$ MPa $\cong 2 \cdot S_{Yc}$: $\Delta\sigma_{EP} = 648$ MPa, $\Delta\varepsilon_{EP} = 1.85\%$ and $N_{EP} = 958$ cycles is the EP prediction, while the LE prediction is $\Delta\sigma_{LE} = 525$ MPa, $\Delta\varepsilon_{LE} = 0.85\%$ and $N_{LE} = 4590$ cycles. For this higher load, the ratio $N_{EP}/N_{LE} = 0.209$ is not negligible at all.

Furthermore, the LE approximation of Neuber's equation can result in physically impossible predictions. For instance, let's estimate the residual stress left at the notch tip of a component with $K_t = 2.9$ by the nominal stress history $\sigma_n = \{0 \to 800 \to 0\}$ MPa (away from the notch), assuming Ramberg–Osgood cyclic parameters $h_c = 0.18$ and $H_c = 1280$ MPa. If the stress σ_1 induced at the tip by the nominal $\sigma_{n1} = 800$ MPa is assumed LE, then by Neuber:

$$\sigma_1^2 + 1.1 \cdot 10^{-12}\sigma_1^{6.56} = (2.9 \cdot 800)^2 \Rightarrow \sigma_1 = 700 \text{ MPa} < \sigma_n ?! \tag{5}$$

Even though this calculation is *mathematically* correct, its prediction is clearly absurd, since the stress at the notch tip cannot be smaller than the nominal stress. In other words, values $K_\sigma < 1$ are meaningless and indicate the model used to calculate them is based on wrong assumptions. On the other hand, when the first load event σ_{n1} is modeled as EP, the notch-tip stress is much more reasonably estimated by:

$$K_t^2\left(\frac{\sigma_{n1}^2}{E} + \frac{\sigma_{n1}^{1+1/h_c}}{H_c^{1/h_c}}\right) = \frac{\sigma_1^2}{E} + \frac{\sigma_1^{1+1/h_c}}{H_c^{1/h_c}} \Rightarrow \sigma_1 = 1116 \text{ MPa} \tag{6}$$

From this value, $\sigma_{res} = \sigma_1 - \Delta\sigma$ is obtained after calculating the unloading loop range:

$$K_t^2\left(\Delta\sigma_n^2 + \frac{2E\Delta\sigma_n^{1+1/h_c}}{(2H_c)^{1/h_c}}\right) = \Delta\sigma^2 + \frac{2E\Delta\sigma^{1+1/h_c}}{(2H_c)^{1/h_c}} \Rightarrow \Delta\sigma = 1256$$
$$\Rightarrow \sigma_{res} = -140 \text{ MPa} \tag{7}$$

These calculations make sense but, as they are based on estimated K_σ and K_ε values, their accuracy can only be properly checked experimentally, or else verified by suitable numeric modeling techniques. However this is easier said than done, since neither such tests nor EP FE calculations are trivial, so they cannot be afforded in most practical applications.

3. Errors Induced by the linear elastic modeling of the nominal stresses

The examples presented in the previous section justify a deeper study of the errors that may be produced by the traditional practice of modeling EP notch effects by Neuber and Ramberg–Osgood, whereas assuming LE nominal stresses if $\boldsymbol{\Delta\sigma_n < 2S_Y}$. Indeed, at least from the logical point of view this practice is inconsistent, as the material in the nominal and critical regions is the same, hence it should be described by only one EP constitutive equation.

To evaluate the importance of this problem in real-life applications, notch tip stress ranges $\Delta\sigma$ and the corresponding lives N are calculated for several geometric stress concentration factors K_t and various nominal stress-to-cyclic-yield strength ratios $\Delta\sigma_n/2S_{Yc}$, using measured properties from 517 steels. The EP formulation uses Eq. (2), while the LE assumes Hooke's law for the nominal stresses. As shown in Table 1, the LE $\Delta\sigma_n$ hypothesis can produce severely *non*-conservative predictions compared to the EP formulation, even for surprisingly small nominal stress ranges.

Moreover, Figs. 2–4 show that the LE formulation can produce absurd predictions for large $\Delta\sigma_n/2S_{Yc}$ ratios, like $K_\sigma < 1 \Rightarrow \Delta_\sigma < \Delta\sigma_n$, or critical stress ranges at the notch tip *smaller* than the nominal ranges, as observed in the previous section. On the other hand, no such problems occur if the nominal ranges are modeled as EP using Eq. (2). This equation predicts that both K_σ and K_ε tend to constant values when the ratio $\Delta\sigma_n/2S_{Yc}$ grows, calculated neglecting the Ramberg–Osgood elastic term at very large plastic strains.

The minimum value that the stress concentration factor K_σ can reach, according to the consistent EP formulation for the Neuber system, occurs when the elastic ranges are negligible:

$$K_t^2\left[\frac{2E\Delta\sigma_n^{(h_c+1)/h_c}}{(2H_c)^{1/h_c}}\right] = \frac{2E\Delta\sigma^{(h_c+1)/h_c}}{(2H_c)^{1/h_c}} \Rightarrow K_{\sigma,\min} = \frac{\Delta\sigma}{\Delta\sigma_n} = K_t^{2h_c/(1+h_c)} \tag{8}$$

Since by Neuber's rule $K_t^2 = K_\sigma K_\varepsilon$, the maximum EP strain concentration factor calculated by it is $K_{\varepsilon,\max} = K_t^2/K_t^{2h_c/(1+h_c)} = K_t^{2/(1+h_c)}$, see Fig. 5. Notice in Fig. 5 that the EP formulation adopts K_σ and K_ε defined for EP nominal stresses and strains, while the LE formulation defines them with respect to LE values. Hence, it is easy to show that the K_σ and K_ε predicted by Neuber (if they are defined based on EP nominal stresses and strains) should be limited by the bounds:

$$\begin{cases} K_t^{2h_c/(1+h_c)} \leqslant K_\sigma \leqslant K_t \\ K_t \leqslant K_\varepsilon \leqslant K_t^{2/(1+h_c)} \end{cases} \tag{9}$$

Notice that Eq. (9) is consistent with the limit case of a material that does not strain harden, where $h_c \to 0$. In this case, gross yielding of the net section would make both nominal σ_n and notch-tip σ stresses become equal to the yield strength, making $K_\sigma = \sigma/\sigma_n \to 1$ in the imminence of plastic collapse, which is coherent with the lower bound $K_t^{2h_c/(1+h_c)} = K_t^0 = 1$ from Eq. (9) for $h_c = 0$.

Large $\Delta\sigma_n$ values are not so common in practice, since they are usually associated with very short lives. Therefore, it might be justifiable to adopt Hooke's law if $\Delta\sigma_n/2S_{Yc} \ll 1$, if nominal stresses are below the elastic limit of the material. Ramberg–Osgood's equation, on the other hand, is not able to consider a purely elastic region, since it always assumes the presence of plastic strains. Thus, Hooke's law and Ramberg–Osgood are not equivalent even in the elastic region. Modeling the nominal $\Delta\sigma_n$ by Hooke and the notch $\Delta\sigma$ by Ramberg–Osgood is inconsistent, since it could generate numeric errors in notch calculations, especially for low values of K_t where both $\Delta\sigma_n$ and $\Delta\sigma$ would achieve similar values but following different equations that would conflict. Even though Ramberg–Osgood is an approximation that does not exactly describe the purely elastic region, it should be used to model both $\Delta\sigma_n$ and $\Delta\sigma$ to avoid numeric errors. Otherwise, if $\Delta\sigma_n$ is modeled using Hooke's law, then $\Delta\sigma$ should be represented by an

Table 1
Maximum errors (always *non*-conservative!) among 517 steels for the LE $\Delta\sigma_n$ predictions using Neuber's system of equations, with respect to Neuber's EP formulation predictions, for $K_t = 3$ and several nominally LE ratios $\Delta\sigma_n/2S_{Yc}$ [4,5].

$\Delta\sigma_n/2S_{Yc}$	Maximum error in $\Delta\sigma$ (%)	Maximum error in N (%)
0.1	5	27
0.3	11	102
0.5	15	202
0.8	21	411
1.0	23	2081

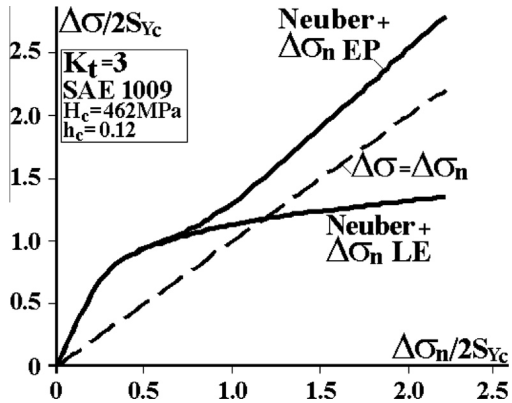

Fig. 2. LE and EP predictions for the maximum stress range $\Delta\sigma$ at a notch tip with $K_t = 3$, for a very ductile SAE 1009 steel. Notice that for very high $\Delta\sigma/2S_{Yc}$ (nominal stress range-to-cyclic yield strength) ratios the LE formulation predicts $\Delta\sigma < \Delta\sigma_n$, which would mean that a notch could *decrease* the overall nominal stresses, a nonsense.

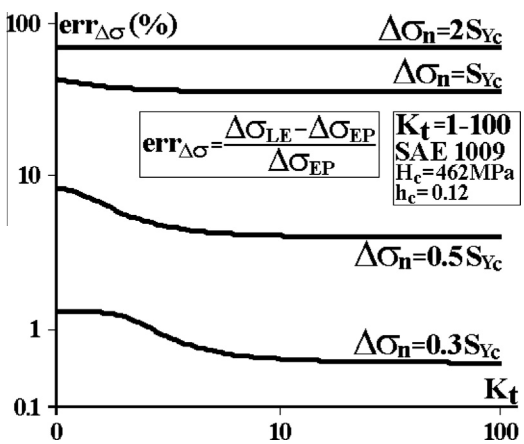

Fig. 3. The errors $err\Delta\sigma = (\Delta\sigma_{LE} - \Delta\sigma_{EP})/\Delta\sigma_{EP}$ in maximum stress ranges predicted assuming the nominal ranges are LE are always non-conservative with respect to the EP predictions. Moreover, these errors are not much dependent on the K_t value.

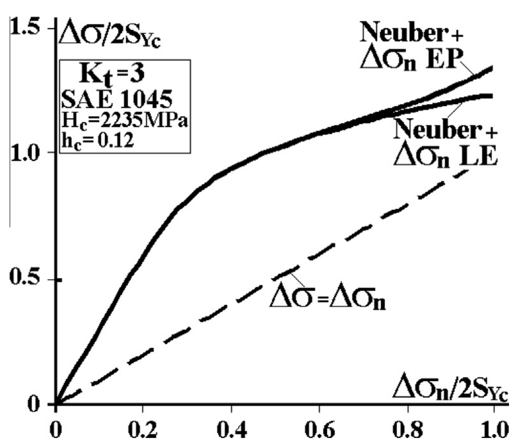

Fig. 4. For very hard quenched 1045 steels, the prediction errors induced by the LE formulation for the Neuber system become almost irrelevant because, due to their very low ductility, a notch with $K_t = 3$ would cause rupture under $\Delta\sigma_n/2S_{Yc} < 1.0$.

elastoplastic equation that also includes a purely elastic region, as opposed to Ramberg–Osgood's. The key issue is to use coherent equations for both nominal and notch regions, to avoid numeric errors.

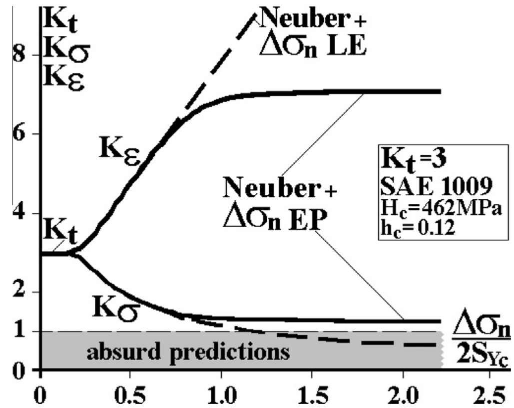

Fig. 5. Unlike the usual LE σ_n formulation, which may lead to absurd predictions, the EP σ_n formulation in Eq. (2) predicts that both the stress and the strain concentration factors tend to limit values given by Eq. (9), at very high $\Delta\sigma_n/2S_{Yc}$ ratios.

Note once again that Ramberg–Osgood may not always provide an accurate representation of the stress–strain behavior, even in the purely elastic region, where it always assumes the presence of (very small) plastic strains. Even though Hooke's law provides a better description of the elastic part, its use mixed with Ramberg–Osgood can lead to the previously presented errors.

But the numeric solution of εN equations is not more difficult when $\Delta\sigma_n$ is also assumed following Ramberg–Osgood. Moreover, gentle notches with $K_t \cong 1$ can have relatively long fatigue lives even under large $\Delta\sigma_n$, and in these cases it is not only possible but also necessary to model the nominal range by Ramberg–Osgood.

However, the EP modeling of nominal ranges involves more complex issues. It is not trivial to calculate nominal EP stresses and strains from a given load history in most practical cases. Indeed, unlike prismatic beams under pure bending or circular shafts under pure torsion, EP nominal strain distributions normally are *not* known beforehand, so it is not possible to use them in equilibrium equations to obtain the corresponding EP stresses. There is no Neuber-like simple rule to contour this problem relating maximum nominal stress and strains either. Moreover, as EP nominal ranges induce memory effects, it is necessary to simultaneously couple load sequence effects in nominal and in notch-tip loops when modeling damage induced by variable amplitude service loads. This is not a simple task. Finally, Neuber's rule may be not be the best estimate even under plane stress conditions, as discussed next.

4. Glinka's strain concentration rule with elastoplastic nominal stresses

Some authors [1] claim that deep notch effects in thick pieces (with t and $a \gg \rho$, where t is the piece thickness, a is the notch depth, and ρ is its tip radius) may be better predicted by the linear strain concentration rule, $\Delta\varepsilon = K_t\Delta\varepsilon_n$. In fact, this rule produced the best results when applied to the critical damage model from [6–10], which uses εN properties to predict fatigue crack propagation rates under dominantly plane-strain conditions, an indication that supports such a claim.

Moreover, Molski and Glinka [11] affirmed Neuber's rule estimates can be too conservative even under plane stress conditions (associated with $t \leqslant \rho$), and proposed an alternative strain concentration estimate supposing the ratio E_D/E_{Dn} between the strain energies associated with notch-tip and with nominal stresses and strains would not be altered by yielding. Therefore, for LE materials, they assumed that

$$\left.\begin{array}{l} E_D = \int_0^{\varepsilon} \sigma(\varepsilon) d\varepsilon = \frac{\sigma^2}{2E} \\ E_{D_n} = \int_0^{\varepsilon} \sigma_n(\varepsilon) d\varepsilon = \frac{\sigma_n^2}{2E} \end{array}\right\} \Rightarrow K_t = \frac{\sigma}{\sigma_n} = \sqrt{\frac{E_D}{E_{D_n}}} \tag{10}$$

If the material follows Ramberg–Osgood, when the notch tip yields under uniaxial loads its strain energy E_D is then given by:

$$\begin{aligned} E_D &= \int_0^{\varepsilon} \sigma(\varepsilon) d\varepsilon = \sigma\varepsilon - \int_0^{\sigma} \varepsilon(\sigma) d\sigma \\ &= \frac{\sigma^2}{E} + \sigma \cdot \left[\frac{\sigma}{H}\right]^{1/h} - \int_0^{\sigma} \left(\frac{\sigma}{E} + \left[\frac{\sigma}{H}\right]^{1/h}\right) d\sigma \Rightarrow E_D \\ &= \frac{\sigma^2}{E} + \sigma \cdot \left[\frac{\sigma}{H}\right]^{1/h} - \frac{\sigma^2}{2E} + \frac{h}{1+h} \left[\frac{\sigma}{H}\right]^{(1+h)/h} \\ &= \frac{\sigma^2}{2E} + \frac{\sigma}{1+h} \left[\frac{\sigma}{H}\right]^{1/h} \end{aligned} \tag{11}$$

where H and h are the monotonic Ramberg–Osgood coefficient and exponent.

Assuming $K_t^2 E_{D_n} = E_D$, the so-called Glinka's rule estimates the EP stress σ at the notch tip from its K_t and from the nominal stress σ_n by:

$$K_t^2 \sigma_n^2 = \sigma^2 + \frac{2E\sigma}{1+h} \left[\frac{\sigma}{H}\right]^{1/h} \tag{12}$$

To apply Glinka's rule to a material following the cyclic $\sigma\varepsilon$ curve, it is enough to switch in the above equation the monotonic Ramberg–Osgood parameters H and h with their cyclic counterparts H_c and h_c. Finally, applying Glinka's rule to a material following the hysteresis loop $\Delta\sigma\Delta\varepsilon$ curve, it is possible to write:

$$(K_t \Delta\sigma_n)^2 = \Delta\sigma^2 + \frac{4E}{(1+h_c)(2H_c)^{1/h_c}} \Delta\sigma^{(h_c+1)/h_c} \tag{13}$$

As $4E/(1 + h_c) < 2E$, the elastoplastic stresses and strains estimated by Glinka at notch tips are always smaller than those estimated by Neuber, and larger than those estimated by the linear strain concentration rule $\Delta\varepsilon = K_t \Delta\varepsilon_n$.

However, Eqs. (12) and (13) intrinsically assume the nominal stresses are LE. As extensively discussed in the previous section regarding Neuber's rule, this hypothesis is only appropriate for nominal stresses σ_n *much* lower than the cyclic yield strength S_{Yc}. Otherwise, strain concentration rules should model both nominal and notch-tip stress–strain relations using Ramberg–Osgood. Under these conditions, Glinka's rule applied to the cyclic $\sigma\varepsilon$ rule and to the loop $\Delta\sigma\Delta\varepsilon$ curve becomes then, respectively,

$$K_t^2 \cdot \left(\sigma_n^2 + \frac{2E\sigma_n}{1+h_c} \left(\frac{\sigma_n}{H_c}\right)^{1/h_c}\right) = \sigma^2 + \frac{2E\sigma}{1+h_c} \left(\frac{\sigma}{H_c}\right)^{1/h_c} \tag{14}$$

$$K_t^2 \cdot \left(\Delta\sigma_n^2 + \frac{4E\Delta\sigma_n}{1+h_c} \left(\frac{\Delta\sigma_n}{2H_c}\right)^{1/h_c}\right) = \Delta\sigma^2 + \frac{4E\Delta\sigma}{1+h_c} \left(\frac{\Delta\sigma}{2H_c}\right)^{1/h_c} \tag{15}$$

Let's emphasize two points about all these simplified procedures: first, EP stress/strain concentration rules are just educated estimates for EP notch effects based on unique LE K_t values. These rules provide *reasonable* but certainly not very precise estimates for K_σ and $K\varepsilon$. Better values can be obtained by numerically calculating EP $\Delta\sigma$ and $\Delta\varepsilon$ ranges using (e.g.) advanced 3D FE techniques, which require erudition and must be used with caution. Indeed, such calculations are neither trivial nor robust. As K_σ and $K\varepsilon$ depend not only on the geometry, but also on how and how much the material strain-hardens, it is certainly better to calibrate them experimentally.

5. Comparison among elastoplastic FE simulations and notch-rule predictions

The presented formulations for Neuber's and Glinka's rules can reasonably estimate the notch behavior under tension and plane stress conditions, even for high nominal stresses that can cause gross yielding of the net section. To verify this, EP Finite Element simulations are performed on a 180 mm-long tensile specimen with a double semi-circular U-notch of radius 24 mm. The specimen height decreases from the nominal 72 mm to $72 - 2 \cdot 24 = 24$ mm at the net cross section. The material is a 304 stainless steel with Young modulus $E = 195$ GPa, Poisson ratio $\nu = 0.29$, yield strength $S_Y = 345$ MPa, and Ramberg–Osgood parameters 556 MPa and 0.08.

Two types of FE simulations are conducted using Abaqus v6.12 software: 2D analyses assuming plane stress conditions, applicable for thin specimens; and 3D analyses for a 100 mm-thick tensile specimen, to approximately simulate plane-strain conditions. The 2D plane stress models were built with 574 quadratic-reduced integration elements (CPS8R from Abaqus library), totalizing 1841 nodes. The 100 mm-thick specimen was modeled with 26,936 quadratic-reduced integration elements (C3D20R from Abaqus library), totalizing 117,481 nodes. In both cases, the modeling took advantage of the symmetry: only one fourth and one eighth of the geometry were modeled in the 2D and 100 mm-thick cases, respectively, and appropriate boundary conditions were applied to the symmetry faces. Also, the respective meshes were selected after cautious convergence studies. The load was applied as a uniform stress in the far-end, and the problem was solved considering small displacements. Fig. 6 shows the expanded symmetric half of the modeled 100 mm-thick double U-notch 3D tensile specimen, along with the adopted FE mesh and coordinate frame. In its net section, the y coordinate varies from -12 to 12 mm (24 mm height), while the thickness coordinate z varies from -50 to 50 mm.

Purely LE FE simulations show that the geometric stress concentration factor K_t for the 2D specimen is equal to 1.31 under tension, with the nominal stress defined in the net section of height 24 mm. On the other hand, for the 100 mm-thick 3D specimens, the critical points are located at the lines $y = \pm 12$ mm, with LE K_t values that vary along the thickness direction z from only 1.06 at the borders $z = \pm 50$ mm up to 1.37 at the mid-thickness $z = 0$, see Fig. 6. Such significant 3D effects had already been studied in the literature [12–14], however they usually are neglected in practical applications. It would not be unusual to perform in practice a 2D FE analysis on the specimen to find its $K_t = 1.31$, and then assume this would be the same value for the

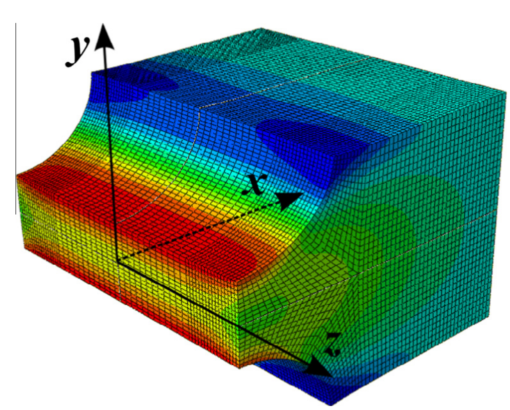

Fig. 6. Symmetric half of the modeled 100 mm-thick double U-notch 3D tensile specimen, along with the adopted FE mesh and coordinate frame.

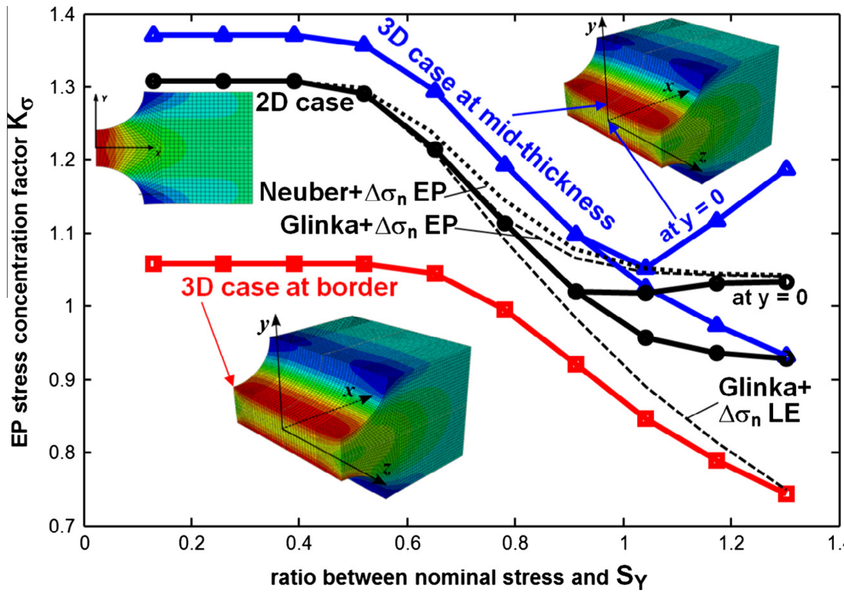

Fig. 7. FE-calculated $K\sigma$ as a function of σ_n/S_Y for the double U-notch tensile specimens, compared with Glinka's and Neuber's rule predictions.

actual 3D component, instead of the almost 5% higher $K_t = 1.37$ at the mid-thickness. Such 3D effects also suggest that a surface fatigue crack would most likely initiate on the thick specimen at the mid-thickness $z = 0$ instead of at the corners from the $z = \pm 50$ mm borders.

Then, EP FE analyses are performed in both 2D and 3D specimens, to find out the EP stress concentration factor K_σ as a function of the applied load level, measured from the ratio σ_n/S_Y, where σ_n is the nominal stress defined at the net section. Fig. 7 shows the FE-calculated K_σ as a function of σ_n/S_Y for the 3D case at the border and at the mid-thickness, as well as for the 2D case. Not surprisingly, the K_σ values for the 2D (thin) specimen are always in-between the mid-thickness and border values from the thick 3D specimen.

Fig. 7 also shows that the K_σ values decrease as the specimen yields, as expected due to stress relief at the notch tip. Moreover, the stresses at the borders of the thick specimen are always smaller than the mid-thickness values. Gross yielding of the net section, which is associated with ratios $\sigma_n/S_Y > 1$, causes the maximum stress to be located below the surface of the specimen, usually at the mid-height $y = 0$, a behavior represented in the graph by bifurcations of the K_σ curves. These bifurcations show the value of K_σ for $y = 0$ and for the surfaces $y = \pm 12$ mm, for each σ_n/S_Y ratio and for each 2D or 3D specimen.

The EP modification of Glinka's notch rule from Eq. (14) is then applied using the $K_t = 1.31$ from the 2D case. As seen in the figure, this rule results in reasonable estimates of the maximum K_σ of the 2D specimen, even for gross yielding of the net section, after the graph bifurcation (where the maximum stress would be located at $y = 0$, instead of at $y = \pm 12$ mm). The EP version of Neuber's rule from Eq. (3), also represented in the figure, results in reasonable but more conservative estimates, slightly overpredicting the K_σ obtained for the 2D case. On the other hand, the classic formulation of Glinka's rule assuming LE nominal stresses would highly underestimate K_σ in the 2D case under large load levels, as shown in Fig. 7. The classic Neuber's rule with LE nominal stresses, not represented in the figure, would also result in poor predictions for high σ_n/S_Y ratios, similar to Glinka's rule without the EP modification. Similar conclusions can be drawn for EP *strain* concentration factors K_ε, which can be highly underestimated using LE-nominal-stress formulations, while the presented EP formulations better estimate the notch-root strains.

Finally, it is important to note that the presented EP corrections of Neuber's and Glinka's rules are only reasonable under tension, or under other loading conditions that do not cause significant gradients in the nominal stress profile. If, on the other hand, nominal bending or torsion causes a stress gradient at the notch root that is not negligible with respect to the notch stress gradient, then both gradients should be considered in the EP correction calculations. One possible approach has been proposed by Seeger and Heuler [15], who corrected the notch equations using the shape factor P_{PC}/P_Y, defined as the ratio between the plastic collapse and yielding loads under bending or torsion, calculated assuming a material that does not strain harden. Such a shape factor is a property of the specimen cross section and of the loading type, and it can be used as a plastic notch factor to define modified Neuber and Glinka rules for high nominal bending and torsion loads. These modifications for bending and torsion are beyond the scope of this work, but they could be implemented without problems in Eqs. (2), (3), (14) and (15).

6. Conclusions

In this work, Neuber's and Glinka's notch rules were evaluated based on Finite Element analyses. It was found that Glinka's rule better estimated the notch-tip stresses and strains under plane stress for the studied double U-notch tensile specimen, when compared to Neuber's. Moreover, Neuber and Glinka formulations considering elastoplastic nominal stresses were presented and evaluated. It can be concluded that nominal stresses should never be assumed as purely elastic in those rules, unless the nominal levels are lower than the elastic limit of the material. Fatigue life calculation errors of up to 2000% could result from the use of such a linear elastic simplification even for nominal stress ranges below the cyclic yield strength. 3D effects on the stress concentration factor were also discussed from the analyses on 2D and thick 3D specimens.

References

[1] R.I. Stephens, A. Fatemi, R.R. Stephens, H.O. Fuchs, Metal Fatigue in Engineering, second ed., Interscience, 2000.

[2] Landgraf,RW. The resistance of metals to cyclic deformation. In: ASTM STP, vol. 467, 1970, pp. 3–36.

[3] H. Neuber, Theory of stress concentration for shear-strained prismatical bodies with an arbitrary non-linear stress-strain law, J. Appl. Mech. 28 (1961) 544–551.
[4] M.A. Meggiolaro, J.T.P, Castro, On the errors induced by the hookean modeling of nominal stresses in the εN method. in: Fatigue, SAE P2001-01-4067, 2001, pp. 257–266.
[5] M.A. Meggiolaro, J.T.P. Castro, Evaluation of the errors induced by high nominal stresses in the classical εN method, in: A.F. Blom (Ed.), Fatigue, EMAS 2002, vol. 2, 2002, pp. 1451–1458.
[6] J.T.P. Castro, P.P. Kenedi, Previsão das taxas de propagação de trincas de fadiga partindo dos conceitos de Coffin-Manson (in Portuguese), J. Braz. Soc. Mech. Sci. Eng. 17 (1995) 292–303.
[7] J.T.P. Castro, M.A. Meggiolaro, A.C.O. Miranda, Singular and non-singular approaches for predicting fatigue crack growth behavior, Int. J. Fatigue 27 (2005) 1366–1388.
[8] J.A.R. Durán, J.T.P. Castro, J.C. Payão Filho, Fatigue crack propagation prediction by cyclic plasticity damage accumulation models, Fatigue Fract. Eng. Mater. Struct. 26 (2003) 137–150.
[9] J.A.R. Durán, J.T.P. Castro, M.A. Meggiolaro, A damage accumulation model to predict fatigue crack growth under variable amplitude loading using εN parameters, in: A.F. Blom (Ed.), Fatigue, EMAS, 2002, vol. 4, 2002, pp. 2759–2776.
[10] J.A.R. Durán, J.T.P. Castro, M.A. Meggiolaro, Crack growth predictions under variable amplitude loading based on low cycle fatigue data, in: P.D. Portella (Ed.), V Low Cycle Fatigue, DVM 2004, pp. 353–358.
[11] K. Molski, G. Glinka, A method of elastoplastic and strain calculation at a notch root, Mater. Sci. Eng. 50 (1981) 93–100.
[12] R.C.O. Góes, J.T.P. Castro, L.F. Martha, Three dimensional effects on elastic notch tip stress/strain fields in finite-thickness plates under tension, in: 67th ABM International Congress, 2012.
[13] Z. Yang, C.B. Kim, C. Cho, H.G. Beom, The concentration of stress and strain in finite thickness elastic plate containing a circular hole, Int. J. Solids Struct. 45 (2008) 713–731.
[14] R.C.O. Góes, J.T.P. Castro, L.F. Martha, 3D effects around notch and crack tips, Int. J. Fatigue 62 (2014) 159–170.
[15] T.H. Seeger, P. Heuler, Generalized application of Neuber's rule, J. Test. Eval. 8 (1980) 199–204.